L-Università ta' Malta

Department of Chemistry
Faculty of Science

B.Sc. (Hons.) Year I

Sample Coursework Assignment

CHE1215: Methods of Chemical Calculations — March 20XX

Instructions

This assignment will assess your knowledge of the topics we have covered so far in class, namely basic algebra, trigonometry, functions, differentiation and integration.

Read the following instructions carefully.

- This assignment carries a small percentage of your final mark for the course.
- This assignment consists of **7 questions** and is out of **100 marks**.
- Submit your solutions in PDF format (handwritten & scanned) on VLE by **XXth of March, 20XX**, no later than 23:55.
- Attempt **all** questions.

⚠ This is not a group assignment: plagiarism will not be tolerated.

MATHEMATICAL FORMULÆ

ALGEBRA

Factors

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

Quadratics

If ax^2+bx+c has roots α and β,

$$\Delta=b^2-4ac$$
$$\alpha+\beta=-\tfrac{b}{a}\qquad \alpha\beta=\tfrac{c}{a}$$

Finite Series

$$\sum_{k=1}^{n}1=n\qquad \sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$
$$\sum_{k=1}^{n}k^2=\frac{k(k+1)(2k+1)}{6}$$
$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^n$$
$$=1+nx+\tfrac{n(n-1)}{2\cdot 1}x^2+\cdots+x^n$$

GEOMETRY & TRIGONOMETRY

Distance Formula

If $A=(x_1,y_1)$ and $B=(x_2,y_2)$,

$$d(A,B)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$
$$=\sqrt{\Delta x^2+\Delta y^2}$$

Pythagorean Identity

$$\cos^2\theta+\sin^2\theta=1$$

General Solutions

$$\cos\theta=\cos\alpha \iff \theta=\pm\alpha+2\pi\mathbb{Z}$$
$$\sin\theta=\sin\alpha \iff \theta=(-1)^n\alpha+\pi n,\ n\in\mathbb{Z}$$
$$\tan\theta=\tan\alpha \iff \theta=\alpha+\pi\mathbb{Z}$$

CALCULUS

Derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x\tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x\cot x$
e^x	e^x
$\log x$	$1/x$
uv	$u'v+uv'$
u/v	$(u'v-uv')/v^2$

Integrals

$f(x)$	$\int f(x)\,dx$
$x^n\ (n\neq-1)$	$\frac{x^{n+1}}{n+1}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$\log(\sec x)$
$\cot x$	$\log(\sin x)$
$\sec x$	$\log(\sec x+\tan x)$
$\operatorname{cosec} x$	$\log(\tan\frac{x}{2})$
e^x	e^x
$1/x$	$\log x$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\frac{x}{\sqrt{a^2+x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$

Homogeneous Linear Second Order ODEs

If the roots of ak^2+bk+c are k_1 and k_2, then the differential equation $ay''+by'+cy=0$ has general solution

$$y(x)=\begin{cases} c_1e^{k_1x}+c_2e^{k_2x} & \text{if } k_1\neq k_2\\ c_1e^{kx}+c_2xe^{kx} & \text{if } k=k_1=k_2\\ e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x) & \text{if } k=\alpha\pm\beta i\in\mathbb{C}\end{cases}$$

Infinite Series

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}\cdots$$
$$\cos x=\sum_{\substack{n=0\\ n\text{ even}}}^{\infty}\frac{(-1)^{\frac{n}{2}}}{n!}x^n=1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots$$
$$\sin x=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n!}x^n=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$
$$\log(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n=x-\frac{x^2}{2}+\frac{x^2}{3}-\cdots,\quad x\in(-1,1]$$

1. A polynomial function f is given by

$$f(x) = 4x^3 - 7x^2 + 7x - 3.$$

   (a) Find the remainder when f is divided by: (i) $3x-1$, (ii) $3x^2-1$.

   (b) Give a full factorisation of f. Hence determine all solutions to

   $$4x^2 + 7 = 7x + \frac{3}{x}.$$

   (c) Decompose the rational function $13/f(x)$ into partial fractions.

   (d) Hence, show that

   $$\int_1^2 \frac{13}{f(x)}\, dx = \log\left(\frac{625}{9}\right) - \frac{\pi}{3\sqrt{3}},$$

   where $\log$ denotes the natural logarithm.

[4, 3, 4, 7 marks]

2. (a) In Hückel molecular orbit theory, the possible values of the orbital energies of the π electrons of ethane (C_2H_4) are given by the stationary values of the quantity

   $$\varepsilon = \alpha + 2\beta c\sqrt{1-c^2},$$

   where α and β are constant 'Hückel parameters', and c is the variable. Show that the possible orbital energies are $\varepsilon = \alpha \pm \beta$.

   (b) Consider the function $f(x, y, z) = x^2z + xy^2 - 2z^3$.

      (i) Determine the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

      (ii) Determine $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial x^2}$.

      (iii) Find the total differential df.

      (iv) Use the total differential to approximate how much the output of f changes when the input goes from $(1, 1, 2)$ to $(1.1, 0.9, 2.2)$.

[4, 9 marks]

3. (a) The volume of a sphere is given by the equation $V = \frac{4}{3}\pi r^3$.

   (i) What is the differential dV in terms of r?

   (ii) The diameter of a ball is measured to be 50 cm, with a margin of error of ± 2 mm. Use the formula above to calculate the volume of the ball, and use the differential to estimate the margin of error.

   (b) Sketch the graphs of the following curves, indicating clearly any asymptotes and intercepts with the coordinate axes.

   (i) $y = 4 - 2\exp(3x - 1)$,

   (ii) $y = \dfrac{x+1}{x-2}$,

   (iii) $y = \cos x$, for $-180° \leqslant x \leqslant 180°$. . **[6, 12 marks]**

4. (a) (i) Sketch the graph of $y = 3\sin(2\vartheta + \frac{\pi}{3})$ for $-2\pi \leqslant \vartheta \leqslant 2\pi$.

   (ii) Solve the trigonometric equation $3\sin(2\vartheta + \frac{\pi}{3}) = \frac{3}{2}$ for all solutions in the range $-2\pi \leqslant \vartheta \leqslant 2\pi$.

   (iii) Indicate your solutions of the equation in (b) in the diagram of part (a).

   (iv) Sketch the unit circle $x^2 + y^2 = 1$, and indicate on it the solutions of the equation $\cos\vartheta = \frac{\sqrt{2}}{2}$ for $\vartheta \in [-2\pi, 2\pi]$.

   (b) Work out $\tan^{-1}(\tan 4)$, and explain why the answer *isn't* 4.

[11, 3 marks]

5. (a) Find the following integrals.

   (i) $\displaystyle\int \frac{dx}{3+x^2}$, (ii) $\displaystyle\int \frac{x}{3+x^2}\,dx$, (iii) $\displaystyle\int \frac{x^2}{3+x^2}\,dx$.

   (b) Sketch $y = x^2+1$ and $y = 4+x-x^2$ on the same axes, and determine the area enclosed between the two curves.

[6, 7 marks]

6. (a) The barometric formula

$$p = p_0\, e^{-Mgh/RT}$$

gives the pressure of a gas of molar mass M at altitude h, when p_0 is the pressure at sea level. Express T in terms of the other variables.

(b) The Clausius–Clapeyron equation for liquid–vapour equilibrium is

$$\frac{d(\log p)}{dT} = \frac{\Delta H_{\text{vap}}}{RT^2}.$$

If the enthalpy of vaporisation, ΔH_{vap}, is constant in the temperature range $T_1 \leqslant T \leqslant T_2$, show that

$$\log\left(\frac{p(T_2)}{p(T_1)}\right) = \frac{\Delta H_{\text{vap}}}{R}\left(\frac{1}{T_1} - \frac{1}{T_2}\right).$$

[4, 6 marks]

7. (a) The rate law of the reaction A $\longrightarrow$ P following first order kinetics with respect to $[A]$ is given by

$$-\frac{d[A]}{dt} = k[A],$$

where $[A]$ is the concentration of A at time t after the commencement of the reaction, and k is the rate constant. If at $t = 0$, $[A] = [A]_0$, show that

$$[A] = [A]_0 \exp(-kt).$$

(b) The Schrödinger equation for a particle inside a one-dimensional box is given by

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi,$$

where $\hbar$, m and E have their usual meaning and can be treated as constants for this question. Show that this has general solution

$$\psi(x) = c_1 \cos\left(\sqrt{\frac{2mE}{\hbar^2}}\, x\right) + c_2 \sin\left(\sqrt{\frac{2mE}{\hbar^2}}\, x\right).$$

[6, 8 marks]

Detailed Solutions

Here I give all solutions and provide detailed explanations; you wouldn't have to explain this much when you provide your solutions. Obviously I do expect you to "show your working" adequately though, and to mention important theorems and facts that you rely on by name.

1. (a) (i) By the remainder theorem, the remainder when f is divided by $(x-\alpha)$ is $f(\alpha)$. For the purposes of the theorem, the factor $3x-1$ is equivalent to $x-\frac{1}{3}$, so the remainder is just $f(\frac{1}{3}) = -\mathbf{\frac{35}{27}}$.

   (ii) Unfortunately the remainder theorem only applies to linear factors, so for this one we have to actually do the division to find the remainder:

$$4x^3 - 7x^2 + 7x - 3 = (3x^2 + 0x - 1)(\tfrac{4}{3}x - \tfrac{7}{3}) + \tfrac{25}{3}x - \tfrac{16}{3}$$

$$\begin{array}{cccccccc}
4x^3 & + & 0x^2 & - & \frac{4}{3}x & & & \\
 & - & 7x^2 & + & 0x & + & \frac{7}{3} & \\
 & & & + & \frac{25}{3}x & - & \frac{16}{3} & \leftarrow \text{remainder} \\
\hline
\checkmark & & \checkmark & & \checkmark & & \checkmark &
\end{array}$$

   Thus the remainder is $\mathbf{\frac{1}{3}(25x-16)}$.

   (b) By the rational roots theorem, any rational roots p/q must have numerator p being a divisor of the constant term -3, and denominator q being a divisor of the leading term 4. Thus any rational root of f must be among $x = \pm\frac{\text{divisors of 3}}{\text{divisors of 4}} = \pm\frac{1,3}{1,2,4}$.

   Trying different values:

$$\begin{array}{ccc}
f(1) = 1 & f(\frac{1}{2}) = -\frac{3}{4} & f(\frac{1}{4}) = -\frac{13}{8} \\
f(-1) = -21 & f(-\frac{1}{2}) = -\frac{35}{4} & f(-\frac{1}{4}) = -\frac{21}{4} \\
f(3) = 63 & f(\frac{3}{2}) = \frac{21}{4} & f(\frac{3}{4}) = 0 \\
f(-3) = -195 & f(-\frac{3}{2}) = -\frac{171}{4} &
\end{array}$$

   and finally, we find that $x = 3/4$ is a rational root,* which corre-

*Remember you can use your digital calculator's variables to speed up this process.

sponds to the fact that $(4x-3) \mid f$. Performing the division,

$$4x^3 - 7x^2 + 7x - 3 = (4x-3)(x^2 - x + 1) + 0$$

$$\begin{array}{ccccccc} 4x^3 & - & 3x^2 & & & & \\ & - & 4x^2 & + & 3x & & \\ & & & + & 4x & - & 3 \\ & & & & & + & 0 \quad \leftarrow \text{remainder} \\ \hline \checkmark & & \checkmark & & \checkmark & & \checkmark \end{array}$$

so we get that $f(x) = (4x-3)(x^2-x+1)$. Now the remaining quadratic factor doesn't have any factors, since its discriminant $\Delta = (-1)^2 - 4(1)(1) = -3$, which isn't a square.[†] Thus the full factorisation of f is $\mathbf{(4x-3)(x^2-x+1)}$.

Now, the given equation easily rearranges into $f(x) = 0$, so the equation is actually just $(4x-3)(x^2-x+1) = 0$. Since $\Delta < 0$, the quadratic factor has no roots, so the only solution to the equation is $\boldsymbol{x = \frac{3}{4}}$.

(c) The partial fraction decomposition of $13/f(x)$ has the shape

$$\frac{13}{(4x-3)(x^2-x+1)} = \frac{A}{4x-3} + \frac{Bx+C}{x^2-x+1},$$

where A, B and C are appropriate constants we need to determine. Clearing denominators, we have

$$13 = A(x^2-x+1) + (Bx+C)(4x-3),$$

and now we can plug in different values of x to determine what the constants should be.

$x = \frac{3}{4} \implies 13 = A(\frac{9}{16} - \frac{3}{4} + 1) + 0 \implies \frac{13}{16}A = 13 \implies A = 16.$
$x = 0 \implies 13 = A - 3C \implies 13 = 16 - 3C \implies C = 1.$
$x = 1 \implies 13 = A + B + C \implies 13 = 16 + B + 1 \implies B = -4.$

Thus, we have

$$\boldsymbol{\frac{13}{f(x)} = \frac{16}{4x-3} + \frac{1-4x}{x^2-x+1}}.$$

[†] In fact, since Δ is negative, it doesn't have any roots either.

(d) Here we use the partial fraction decomposition to obtain

$$
\begin{aligned}
\int_1^2 \frac{13}{f(x)}\,dx &= 16\int_1^2 \frac{dx}{4x-3} + \int_1^2 \frac{1-4x}{x^2-x+1}\,dx \\
&= 16\left[\frac{\log(4x-3)}{4}\right]_1^2 - 2\int_1^2 \frac{2x-\frac{1}{2}}{x^2-x+1}\,dx \\
&= 4\log 5 - 2\int_1^2 \frac{2x-1+\frac{1}{2}}{x^2-x+1}\,dx \\
&= 4\log 5 - 2\int_1^2 \frac{2x-1}{x^2-x+1}\,dx - \int_1^2 \frac{dx}{x^2-x+1} \\
&= 4\log 5 - 2\left[\log(x^2-x+1)\right]_1^2 - \int_1^2 \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} \\
&= 4\log 5 - 2\log 3 - \int_1^2 \frac{dx}{(x-\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \log\left(\frac{625}{9}\right) - \left[\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right)\right]_1^2 \\
&= \log\left(\frac{625}{9}\right) - \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_1^2 \\
&= \log\left(\frac{625}{9}\right) - \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) \\
&= \log\left(\frac{625}{9}\right) - \frac{\pi}{3\sqrt{3}},
\end{aligned}
$$

as required. Notice that we made use of the affine argument rule repeatedly, logarithmic derivatives, and completing the square in the denominator. □

2. (a) Stationary values are the values that a function takes on at the points where its derivative is zero. In this case, the derivative is

$$\begin{aligned} \varepsilon'(c) &= 2\beta\sqrt{1-c^2} + 2\beta c \cdot \frac{1}{2\sqrt{1-c^2}} \cdot (-2c) \\ &= \frac{2\beta(1-2c^2)}{\sqrt{1-c^2}} \end{aligned}$$

by the product rule, and it's not hard to see that this is zero when $c = \pm\frac{1}{\sqrt{2}}$. Thus the stationary values are $\varepsilon(\frac{1}{\sqrt{2}}) = \alpha + \beta$ and $\varepsilon(-\frac{1}{\sqrt{2}}) = \alpha - \beta$, and these are (by what we are told in the question) the possible orbital energies.

(b) Recall that when taking partial derivatives, we treat all other variables as if they are constants. Thus,

$$\frac{\partial f}{\partial x} = \mathbf{2xz + y^2}, \qquad \frac{\partial f}{\partial y} = \mathbf{2xy}, \qquad \frac{\partial f}{\partial z} = \mathbf{x^2 - 6z^2}.$$

(c) For the mixed partial derivative, we can either differentiate $\frac{\partial f}{\partial x}$ with respect to y, or $\frac{\partial f}{\partial y}$ with respect to x. Although not obvious, these will always be the same.

$$\frac{\partial^2 f}{\partial x \partial y} = \mathbf{2y}, \qquad \frac{\partial^2 f}{\partial x^2} = \mathbf{2z}.$$

(d) The total differential of f is

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \mathbf{(2xz + y^2)\,dx + 2xy\,dy + (x^2 - 6z^2)\,dz}, \end{aligned}$$

with reference to part (b).

(e) When the input changes from $(1, 1, 2)$ to $(1.1, 0.9, 2.2)$, we have $dx = 0.1$, $dy = -0.1$ and $dz = 0.2$. Thus, since we are starting from $(1, 1, 2)$, the change in f is approximately

$$\begin{aligned} df &= (2 \cdot 1 \cdot 2 + 1^2) \cdot 0.1 + 2 \cdot 1 \cdot 1 \cdot (-0.1) + (1^2 - 6 \cdot 2^2) \cdot 0.2 \\ &= \mathbf{-4.3}. \end{aligned}$$

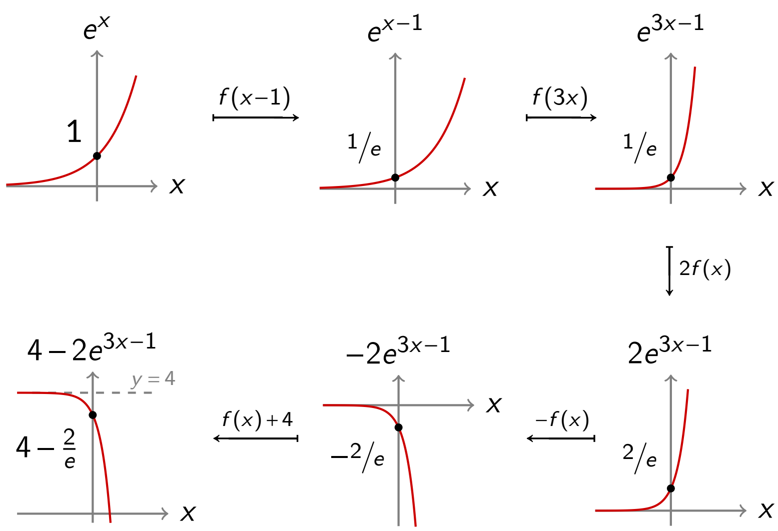
3. (a) (i) $dV = V'(r)\,dr = \boldsymbol{4\pi r^2\,dr}$.

(ii) If the diameter is 50 ± 0.2 cm, then the radius is 25 ± 0.1 cm. Thus the volume is $V = \frac{4}{3}\pi \cdot 25^3 \approx 65\,449.8\,\text{cm}^3$, and for the margin of error in this calculation, we have $dV = 4\pi \cdot 25^2 \cdot 0.1 \approx 785.3\,\text{cm}^3$.

Thus the volume is $\mathbf{65\,449.8 \pm 785.3\,cm^3}$.

(b) Recall affine transformations of functions, and how each of them correspond to graphical steps we can apply to the standard graphs of elementary functions to obtain graphs of more complicated functions.‡

(i) Here is one possible sequence of steps:

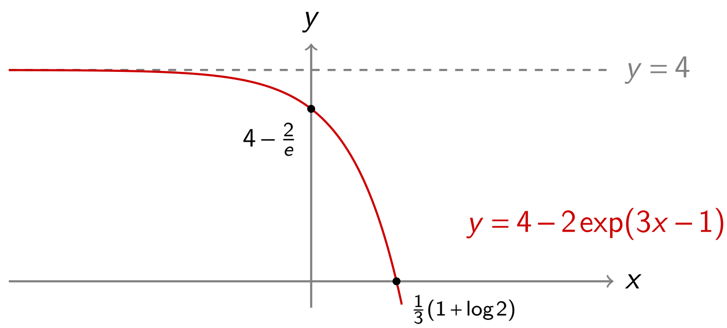


Notice the y-intercept is already done since we kept track of it along the way (alternatively, we can just put $x = 0$). For the x-intercept, we set $y = 0$:

$$\begin{aligned} y = 0 \implies 4 - 2e^{3x-1} = 0 &\implies e^{3x-1} = 2 \\ &\implies 3x - 1 = \log 2 \\ &\implies x = \tfrac{1}{3}(1 + \log 2). \end{aligned}$$

‡Online lecture 1 on VLE and the in-person lecture just before that one are helpful for these.

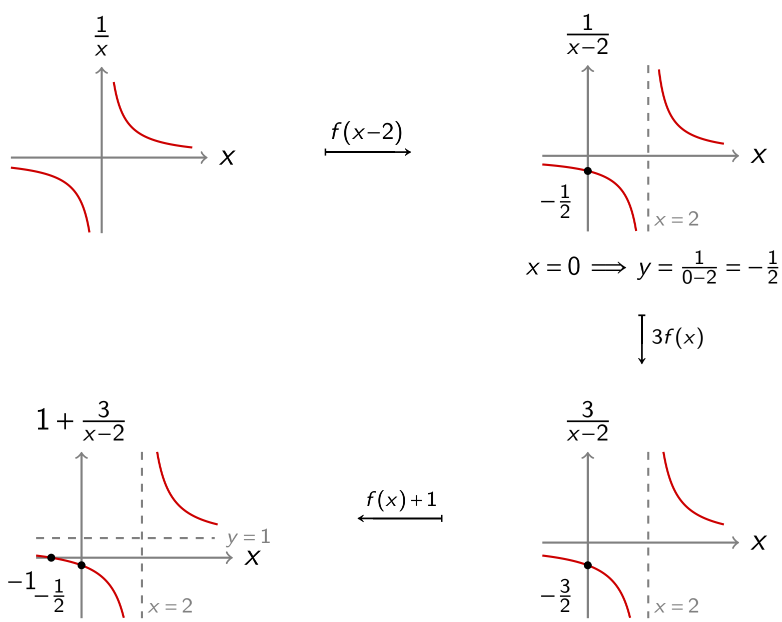
Final sketch:



(ii) Notice that this is an improper fraction, but by polynomial division we can make it proper:

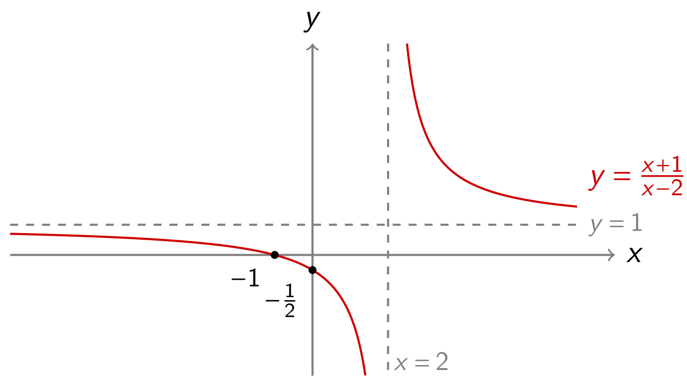
$$y = \frac{x+1}{x-2} = \frac{(x-2)(1)+3}{x-2} = 1 + \frac{3}{x-2}.$$

In this shape, we see that this is an affinely transformed version of $\frac{1}{x}$. A possible sequence of transformations is:

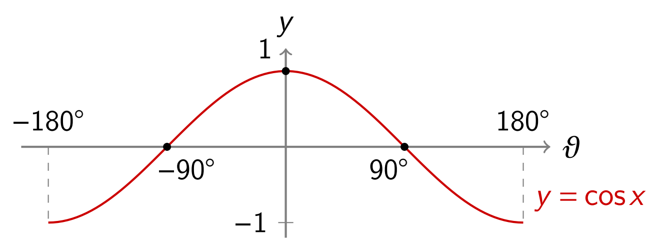


For x-intercept, we put $y = 0 \implies \frac{x+1}{x-2} = 0 \implies x = -1$.

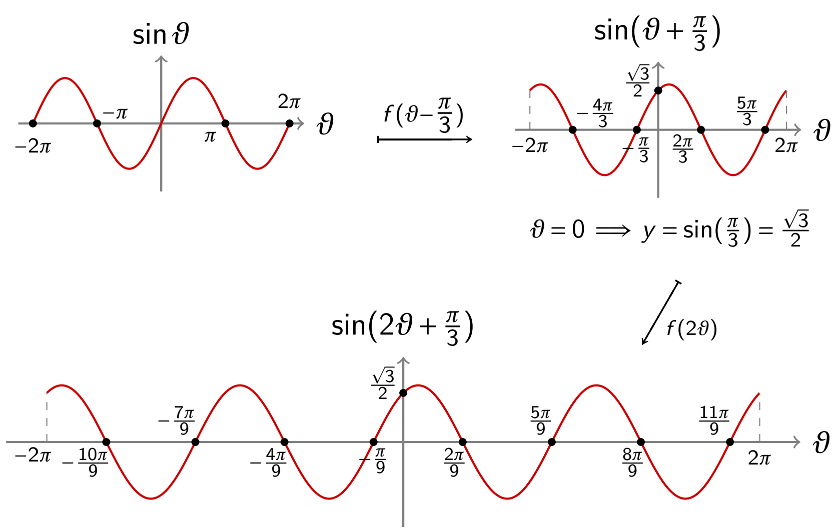
Final sketch:



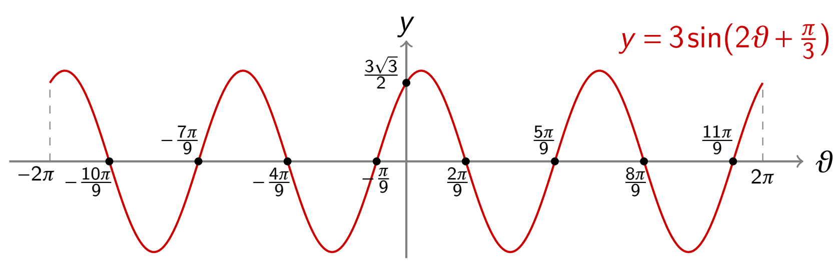
(iii) You should know how to sketch this graph offhand:



4. (a) (i) A possible sequence of steps for this one is:



Thus, scaling everything up by 3, we get the final sketch:



(ii) To solve this equations, we use the usual strategy: find one solution (the *principal value*), and obtain all others using the general solution for the sine function.

$$
\begin{aligned}
& & 3\sin(2\vartheta+\tfrac{\pi}{3}) &= \tfrac{3}{2} & \\
&\implies & \sin(2\vartheta+\tfrac{\pi}{3}) &= \tfrac{1}{2} & \\
&\implies & (2\vartheta+\tfrac{\pi}{3})_{\text{p.v.}} &= \sin^{-1}\left(\tfrac{1}{2}\right) = \tfrac{\pi}{6} & \S \\
&\implies & 2\vartheta+\tfrac{\pi}{3} &= (-1)^n\tfrac{\pi}{6}+n\pi & (n\in\mathbb{Z}) \\
&\implies & 2\vartheta &= (-1)^n\tfrac{\pi}{6}+n\pi-\tfrac{\pi}{3} & \\
&\implies & \vartheta &= (-1)^n\tfrac{\pi}{12}+n\tfrac{\pi}{2}-\tfrac{\pi}{6} & \\
&\therefore & \text{Gen. sol.: } \vartheta &= \tfrac{\pi}{12}(6n+(-1)^n-2) & (n\in\mathbb{Z})
\end{aligned}
$$

Now to get all solutions in the range $-2\pi \leqslant \vartheta \leqslant 2\pi$, we plug in different integer values of n into the general solution.

$$
\begin{aligned}
n=0 &\implies \vartheta = -\tfrac{\pi}{12} \\
n=1 &\implies \vartheta = \tfrac{3\pi}{12} = \tfrac{\pi}{4} \\
n=2 &\implies \vartheta = \tfrac{11\pi}{12} \\
n=3 &\implies \vartheta = \tfrac{15\pi}{12} = \tfrac{5\pi}{4} \\
n=4 &\implies \vartheta = \tfrac{23\pi}{12}
\end{aligned}
$$

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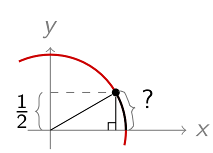
§Now we could just use our calculator to find $\sin^{-1}\left(\frac{1}{2}\right)$, but it's good to remember what it represents: it is the *smallest* angle (i.e., the unique $\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$) which satisfies $\sin\vartheta = \frac{1}{2}$, i.e., the smallest angle whose corresponding point on the unit circle has y-coordinate equal to $\frac{1}{2}$. Doing a rough sketch of the unit circle, we see that that angle is $\frac{\pi}{6}$ (i.e., 30°).

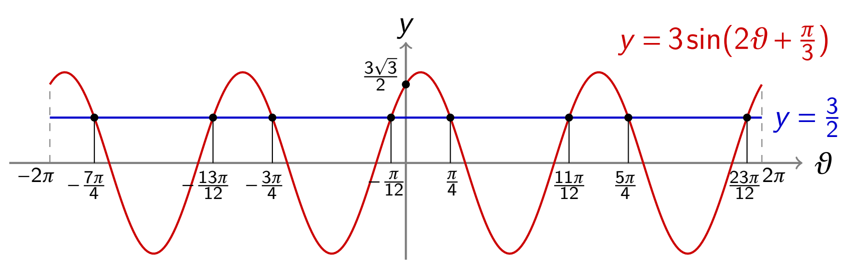


If we plug in $n = 5$ we get a solution which is larger than 2π, so now we move on to negative values of n.

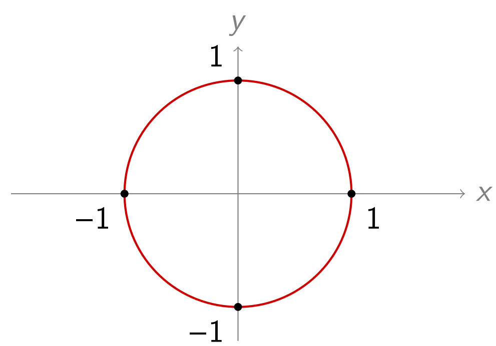
$$
\begin{aligned}
n = -1 &\implies \vartheta = -\tfrac{9\pi}{12} = -\tfrac{3\pi}{4} \\
n = -2 &\implies \vartheta = -\tfrac{13\pi}{12} \\
n = -3 &\implies \vartheta = -\tfrac{21\pi}{12} = -\tfrac{7\pi}{4},
\end{aligned}
$$

and if we take $n = -4$, the corresponding value of ϑ lies outside the range. Thus, gathering everything together, we have the solutions $\boldsymbol{\vartheta} = -\frac{7\pi}{4}, -\frac{13\pi}{12}, -\frac{3\pi}{4}, -\frac{\pi}{12}, \frac{\pi}{4}, \frac{11\pi}{12}, \frac{5\pi}{4}, \frac{23\pi}{12}$.

(iii) The solutions of $3\sin(2\vartheta + \frac{\pi}{3}) = \frac{3}{2}$ correspond to where the curve $y = 3\sin(2\vartheta + \frac{\pi}{3})$ and the line $y = \frac{3}{2}$ intersect, i.e.,



(iv) Here is the sketch:



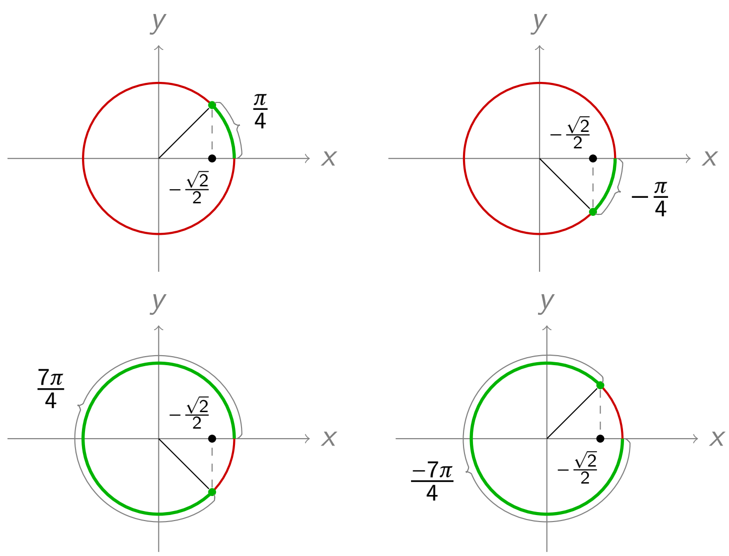
Now the equation:

$$
\begin{aligned}
& & \cos\vartheta &= \tfrac{\sqrt{2}}{2} \\
&\implies & \vartheta_{\text{p.v.}} &= \cos^{-1}\left(\tfrac{\sqrt{2}}{2}\right) = \tfrac{\pi}{4} && \P \\
&\therefore & \vartheta &= \pm\tfrac{\pi}{4} + 2\pi n && (n \in \mathbb{Z})
\end{aligned}
$$

Substituting different values of n,

$$
\begin{aligned}
n = 0 &\implies \vartheta = \pm\tfrac{\pi}{4} \\
n = 1 &\implies \vartheta = \tfrac{7\pi}{4} \text{ or } \tfrac{9\pi}{4} \quad \text{(out of range)} \\
n = -1 &\implies \vartheta = -\tfrac{7\pi}{4} \text{ or } -\tfrac{9\pi}{4} \quad \text{(out of range)}
\end{aligned}
$$

and these are the only n for which the solutions are in range, thus the solutions are $\boldsymbol{\vartheta} = \pm\boldsymbol{\frac{\pi}{4}}, \pm\boldsymbol{\frac{7\pi}{4}}$. On the circle, these angles are:



(b) The tangent function is periodic, with period π. It is therefore not injective. Because of this, the *inverse* function $\tan^{-1}$ is defined to return a principle value in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Thus, $\tan^{-1}(\tan x)$ is always the unique angle ϑ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\tan\vartheta = \tan x$. Clearly $4 \notin (-\frac{\pi}{2}, \frac{\pi}{2})$, but $4-\pi$ is in $(-\frac{\pi}{2}, \frac{\pi}{2})$, and because of periodicity, $\tan(4-\pi) = \tan(4)$.

It follows that $\tan^{-1}(\tan 4)$ is $\boldsymbol{4 - \pi}$.

[¶]Think about how to find this, just as discussed in footnote §. This time, being a cosine, it's the x-coordinate instead.

5. (a) (i) Since $3 = (\sqrt{3})^2$, we can just use the standard formula for this one:

$$\int \frac{dx}{3+x^2} = \mathbf{\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c}.$$

(ii) For this one, we can spot the logarithmic derivative:

$$\begin{aligned}\int \frac{x}{3+x^2}\,dx &= \frac{1}{2}\int \frac{2x}{3+x^2}\,dx \\ &= \tfrac{1}{2}\log(3+x^2) + c \\ &= \mathbf{\log\sqrt{3+x^2} + c}.\end{aligned}$$

(iii) Here we should spot that the fraction is *improper*:

$$\begin{aligned}\int \frac{x^2}{3+x^2}\,dx &= \int \frac{3+x^2-3}{3+x^2}\,dx \\ &= \int \left(1 - \frac{3}{3+x^2}\right) dx \\ &= \int dx - 3\int \frac{dx}{3+x^2} \\ &= \mathbf{x - \sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c}.\end{aligned}$$

(b) To sketch $y = x^2 + 1$, we can just observe that it's a simple affine transformation of $y = x^2$ (namely, an upward shift of 1 unit).

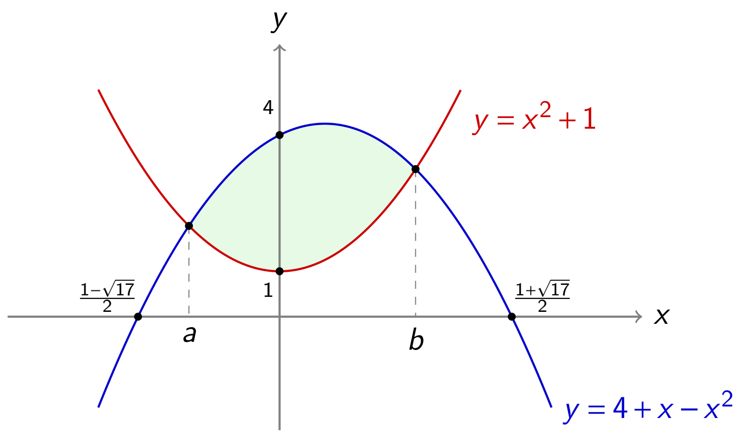
For $y = 4 + x - x^2$, we have a few options. We can first try to find the x- and y-intercepts.

For y-intercepts, we put $x = 0 \implies y = 4$.

For x-intercepts, we put $y = 0 \implies x^2 - x - 4 = 0$

$$\begin{aligned}&\implies (x - \tfrac{1}{2})^2 - \tfrac{1}{4} - 4 = 0 \quad \text{(CTS)} \\ &\implies (x - \tfrac{1}{2})^2 = \tfrac{17}{4} \\ &\implies x - \tfrac{1}{2} = \pm\tfrac{\sqrt{17}}{2} \\ &\therefore \quad x = \tfrac{1\pm\sqrt{17}}{2}.\end{aligned}$$

Noting that these two values are approximately -1.56 and 2.56 respectively, and that the leading coefficient of $4+x-x^2$ is $a=-1$ (so the turning point is a maximum), we can produce a decent sketch of the desired area:



Now, notice that the desired area is

 = −

$$= \int_a^b (4+x-x^2)\,dx - \int_a^b (x^2+1)\,dx$$
$$= \int_a^b (3+x-2x^2)\,dx,$$

where a and b are the x-coordinates of the points of intersection of the two curves. The two curves intersect when their y-coordinates are equal for the same x, i.e., when

$$\begin{aligned} & x^2+1 = 4+x-x^2 \\ \Longrightarrow\quad & 2x^2-x-3=0 \\ \Longrightarrow\quad & (x+1)(2x-3)=0 \\ \Longrightarrow\quad & x=-1 \quad \text{or} \quad x=\tfrac{3}{2}, \end{aligned}$$

i.e., $a = -1$ and $b = \frac{3}{2}$, and so the desired area is

$$\int_{-1}^{3/2} (3 + x - 2x^2)\, dx$$
$$= \left[3x - \frac{x^2}{2} - \frac{2x^3}{3}\right]_{-1}^{3/2}$$
$$= \left(3(\tfrac{3}{2}) - \frac{(\frac{3}{2})^2}{2} - \frac{2(\frac{3}{2})^3}{3}\right) - \left(3(-1) - \frac{(-1)^2}{2} - \frac{2(-1)^3}{3}\right)$$
$$= \mathbf{\frac{125}{24}}.$$

6. (a) This is just a simple "subject of the formula" question, where we need to solve for the power (and therefore we'll have to use logs).

$$p = p_0 e^{-Mgh/RT}$$
$$\implies \quad e^{-Mgh/RT} = \frac{p}{p_0}$$
$$\implies \quad -\frac{Mgh}{RT} = \log\left(\frac{p}{p_0}\right)$$
$$\therefore \quad \boldsymbol{T} = -\frac{\boldsymbol{Mgh}}{\boldsymbol{R}\log\left(\boldsymbol{p}/\boldsymbol{p_0}\right)}\ .$$

(b) This is a simple differential equation, all we have to do is integrate both sides with respect to T. Since T_1 and T_2, the range of values of T, are appearing in the equation we wish to prove, it suggests we should integrate over that range:

$$\int_{T_1}^{T_2} \frac{d(\log p)}{dT}\, dT = \int_{T_1}^{T_2} \frac{\Delta H_{\text{vap}}}{RT^2}\, dT$$
$$\implies \quad [\log p]_{T_1}^{T_2} = \frac{\Delta H_{\text{vap}}}{R} \int_{T_1}^{T_2} \frac{dT}{T^2}$$
$$\implies \quad \log(p(T_2)) - \log(p(T_1)) = \frac{\Delta H_{\text{vap}}}{R} \left[-\frac{1}{T}\right]_{T_1}^{T_2}$$
$$\therefore \quad \log\left(\frac{p(T_2)}{p(T_1)}\right) = \frac{\Delta H_{\text{vap}}}{R}\left(\frac{1}{T_1} - \frac{1}{T_2}\right) \qquad \square$$

7. (a) This is a separable first order ordinary differential equation. Indeed, we can separate the variables:

$$
\begin{aligned}
& & -\frac{d[A]}{dt} &= k[A] \\
&\implies & \frac{d[A]}{[A]} &= -k\,dt \\
&\implies & \int \frac{d[A]}{[A]} &= -k \int dt \\
&\implies & \log[A] &= -kt + \log c \\
&\implies & [A] &= \exp(-kt + \log c) \\
&\therefore & \text{Gen. sol.:}\quad [A] &= c\,\exp(-kt).
\end{aligned}
$$

Since we are given that when $t = 0$, $[A] = [A]_0$, we have that

$$[A]_0 = c\exp(-k \cdot 0) \implies [A]_0 = c,$$

and thus we have the particular solution

$$[A] = [A]_0 \exp(-kt),$$

as required. □

(b) This is a homogeneous second order linear ordinary differential equation with constant coefficients. The first step is to rearrange it into the form

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + E\psi = 0,$$

so we have the auxiliary equation

$$
\begin{aligned}
& & \frac{\hbar^2}{2m}k^2 + E &= 0 \\
&\implies & k^2 &= -\frac{2mE}{\hbar^2} \\
&\implies & k &= \pm\sqrt{\frac{2mE}{\hbar^2}}\,i,
\end{aligned}
$$

and so the general solution is that of the case of complex roots, i.e.,

$$\psi(x) = c_1 \cos\left(\sqrt{\frac{2mE}{\hbar^2}}\, x\right) + c_2 \sin\left(\sqrt{\frac{2mE}{\hbar^2}}\, x\right),$$

as required. □