L-Università ta' Malta

Department of Mathematics
Faculty of Science

B.Sc. (Hons.) Year I

Summer Examination Session 2023

MAT1804: Mathematics for Computing — 13th September 2023, 08:30–10:35

Instructions

Read the following instructions carefully.

- Attempt only **THREE** questions.
- Each question carries **35** marks.
- Calculators and mathematical formulæ booklet will be provided.

⚠ Attempt only **THREE** questions.

Question 1.

(a) State and prove the handshaking lemma.

(b) Let T be a tree on n vertices.

(i) Explain why T has at least two leaves.

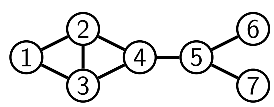
(ii) Show that $|E(T)| = n-1$.

(iii) Suppose that all the vertices are either leaves or have degree d. If there are k vertices of degree d, show that $n = 2 + k(d-1)$.

(c) Let G be a d-regular graph on n vertices. Show that $\chi(G) \geqslant \dfrac{n}{n-d}$.

[*Hint: Consider a $\chi(G)$-colouring of G, and suppose that n_1 vertices get colour 1, n_2 vertices get colour 2, ..., $n_{\chi(G)}$ get colour $\chi(G)$. Then $n = n_1 + \cdots + n_{\chi(G)}$, and argue that $n_i \leqslant n-d$ for all i.*]

(d) Consider the graph G below.



(i) Find its adjacency matrix, (ii) Find its density,

(iii) Draw a 3-colouring of G, and explain why $\chi(G) = 3$.

[9, 10, 8, 8 marks]

Question 2.

(a) You may assume that every rational number can be written in the form a/b where $a, b \in \mathbb{Z}$ and $\text{hcf}(a, b) = 1$.

(i) Show using induction that every natural number n is either odd or even, i.e., we can express $n = 2k$ or $n = 2k+1$ for some $k \in \mathbb{N}$.

(ii) For any integer x, show that if x^3 is even, then x is even.

(iii) Hence, show that $\sqrt[3]{2}$ is irrational.

(iv) Deduce that the number $\dfrac{1+2\sqrt[3]{2}}{1-2\sqrt[3]{2}}$ is irrational.

(b) Using induction, prove that $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$.

(c) Deduce that

$$1+abcd \geqslant \tfrac{1}{8}(1+a)(1+b)(1+c)(1+d)$$

by first proving the more general statement

$$2^{n-1}(1+a_1a_2\cdots a_n) \geqslant (1+a_1)(1+a_2)\cdots(1+a_n)$$

using strong induction.

[15, 8, 12 marks]

Question 3.

(a) Define the function $f:(-1,1)\to\mathbb{R}$ by

$$f(x)=\frac{x}{\sqrt{1-x^2}}.$$

Prove that f is a bijection, hence determine a formula for f^{-1}.

(b) Let A and B be two finite sets.

(i) Show that the number of functions from A to B is $(|B|+1)^{|A|}$.

(ii) How many total injective functions are there from A to B?

(c) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^2+1$. Find:

(i) $f([1,4])$ (ii) $f(\{9\})$ (iii) $f([1,2]\cup(3,4])$

(iv) $f([-3,3])$ (v) $f^{-1}(\{3\})$ (vi) $f^{-1}([-3,3])$

(d) Let $f:X\to Y$ be any function.

(i) Show that for any $A,B\subseteq Y$, we have $f^{-1}(A\cup B)=f^{-1}(A)\cup f^{-1}(B)$.

(ii) Is it always true that $f(A\cup B)=f(A)\cup f(B)$ for $A,B\subseteq X$? If yes, prove it, if no, give a counterexample.

[10, 10, 6, 9 marks]

Question 4.

(a) A total function $f: A \to \mathbb{R}$ is said to be *Lipschitz continuous* if

$$\exists k > 0 : \forall x, y \in A, |f(x) - f(y)| \leqslant k|x - y|.$$

Write down the negation of the above statement.

(b) Construct a truth table for the proposition $(\neg\varphi \wedge \psi) \vee \xi \to \neg\psi \vee \neg\xi$.

(c) Prove that for any two sets A and B, $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

(d) Let M be the set of months in the year 2023, and for $m_1, m_2 \in M$, let $m_1 \sim m_2$ if the 1st of m_1 and the 1st of m_2 are the same day of the week (both Tuesday, for instance).

(i) Show that $\sim$ is an equivalence relation.

(ii) Find its equivalence classes.

(iii) Draw a graph G, with $V(G) = M$, and $E(G) = \{m_1 m_2 : m_1 \sim m_2\}$.

(e) Prove that for all integers a and m, if a and m are the lengths of the sides of a right-angled triangle and $m+1$ is the length of the hypotenuse, then a is an odd integer.

[4, 5, 10, 9, 7 marks]

Answers and Hints

1. (a) Standard from notes.

(b) (i) If we consider a longest path in T, then the end vertices of the path cannot have any neighbours not on the path, since then the path wouldn't be longest. They cannot have any other neighbours on the path either, since then T wouldn't be a tree. Therefore they cannot have any neighbours, i.e., they must be leaves. □

(ii) By induction on n. Clearly when $n=1$ we have $0=n-1$ edges, which establishes the base case. Now given a tree T on n vertices, remove a leaf ℓ (we know there are always at least 2 leaves by (i)) to get $T-\ell$, which by the IH has $(n-1)-1=n-2$ edges. But adding ℓ back increases the number of edges by 1, so we have $n-1$ edges. □

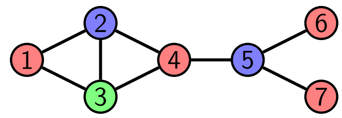
(iii) Since there are k vertices of degree d, then there are $n-k$ vertices of degree 1, and so the sum of degrees is $k\cdot d+1\cdot(n-k)$, which by the handshaking lemma equals $2|E(G)|=2(n-1)$. Solving the equation $kd+(n-k)=2(n-1)$ for n gives the desired result. □

(c) Following the hint, we may write $n=n_1+\cdots+n_\chi$. Since G is d-regular, each vertex v has d neighbours which must be given a distinct colour, thus there are at least d vertices receiving a different colour from v, i.e., at most $n-d$ can have the same colour as v. Thus $n_i \leqslant n-d$ for all i, and so $n=\sum_{i=1}^{\chi} n_i \leqslant \sum_{i=1}^{\chi}(n-d)=\chi(n-d)$.

(d) (i)
$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(ii) $\rho(G)=|E(G)|/\binom{7}{2}=8/21$.

(iii) An example of a 3-colouring:



Clearly G cannot be 2-coloured since it contains an odd cycle and is therefore not bipartite. Thus $\chi(G) > 2$. Since we have provided a 3-colouring, $\chi(G) \leqslant 3$. Thus $\chi(G) = 3$.

2. (a) (i) For the base case, $n = 1$ can be expressed as $2 \cdot 0 + 1$.

Now for the inductive step, suppose $n-1$ can be expressed as $2k$ or $2k+1$. In the first case, we have $n = 2k+1$, and in the second case, $n = 2(k+1)$, which completes the proof. □

(ii) By contrapositive: we show that if x is not even, then x^3 is not even. By part (i), this is equivalent to: if x is odd, then x^3 is odd.

Therefore, suppose x is odd. Then we may write it as $x = 2k+1$ for some k. But then $x^3 = (2k+1)^3 = 1 + 6k + 12k^2 + 8k^3 = 2(4k^3 + 6k^2 + 3k) + 1$, which is also clearly odd. □

(iii) By contradiction: suppose $\sqrt[3]{2}$ is rational, so we may express $\sqrt[3]{2} = a/b$ with $\mathrm{hcf}(a, b) = 1$. But then $a^3 = 2b^3$, which means that a^3 is even, and so by part (ii) a is even, say $a = 2k$ for some k.

Then $a^3 = 2b^3 \implies (2k)^3 = 2b^3 \implies 2(2k^3) = b^3$, i.e., b^3 is even, which by (ii) implies that b is even.

Thus a and b are both divisible by 2. This contradicts that $\mathrm{hcf}(a, b) = 1$. □

(iv) If the given number is rational, say equal to a/b with $a, b \in \mathbb{Z}$, then we may make $\sqrt[3]{2}$ subject of the equation to get that $\sqrt[3]{2} = (a-b)/(2a+2b)$, contradicting that $\sqrt[3]{2}$ is irrational.

(b) For the base case, when $n = 1$, we clearly get equality. Now for the inductive step, suppose the statement holds for $n-1$, i.e.,

$$\frac{1}{1 \cdot 2} + \cdots + \frac{1}{(n-1) \cdot n} = \frac{n-1}{n}.$$

Adding $\frac{1}{n(n+1)}$ to both sides gives us that the sum is

$$\frac{n-1}{n}+\frac{1}{n(n+1)}=\frac{n}{n+1},$$

which completes the proof. $\square$

(c) When $n=1$ both sides are equal, so the inequality is trivial. Now for the inductive step,

$$\begin{aligned}2^{n-1}(1+a_1a_2\cdots a_n) &\geqslant (1+a_1)(1+a_2)\cdots 2(1+a_{n-1}a_n) && \text{(by IH)}\\ &\geqslant (1+a_1)(1+a_2)\cdots(1+a_{n-1})(1+a_n) && \text{(by IH)},\end{aligned}$$

as required. Thus taking $n=4, a_1=a, a_2=b, a_3=c, a_4=d$, we get

$$(1+abcd)\geqslant \tfrac{1}{8}(1+a)(1+b)(1+c)(1+d). \qquad \square$$

3. (a) Firstly, f clearly assigns each $x\in(-1,1)$ to a unique real number, so f is total and functional.

Now to see that f is injective, suppose $f(x)=f(y)$. Then:

$$\begin{aligned} & \frac{x}{\sqrt{1-x^2}}=\frac{y}{\sqrt{1-y^2}}\\ \implies\quad & \frac{x^2}{1-x^2}=\frac{y^2}{1-y^2}\\ \implies\quad & x^2(1-y^2)=y^2(1-x^2)\\ \implies\quad & x^2=y^2\\ \implies\quad & y=\pm x\end{aligned}$$

Clearly $f(x)\neq f(-x)$ since they have opposite signs, so we must have $x=y$, thus f is an injection.

Finally, to see that f is a surjection, let $y\in\mathbb{R}$. Then

$$\begin{aligned} & f(x)=y\\ \implies\quad & \frac{x}{\sqrt{1-x^2}}=y\\ \implies\quad & x^2=y^2(1-x^2)\end{aligned}$$

$$\begin{aligned}
\implies \quad & x^2 + x^2y^2 = y^2 \\
\implies \quad & x^2(1+y^2) = y^2 \\
\implies \quad & x = \pm\frac{y}{\sqrt{1+y^2}}
\end{aligned}$$

Of the two, it is clear that we should take the + version, since

$$f\left(\frac{y}{\sqrt{1+y^2}}\right) = \frac{\frac{y}{\sqrt{1+y^2}}}{\sqrt{1-\frac{y^2}{1+y^2}}} = \frac{y}{1-y^2},$$

thus for any $y \in \mathbb{R}$, we can take $x = y/\sqrt{1+y^2}$ to get $f(x) = y$, proving that f is surjective.

Consequently, we see that

$$f^{-1}(x) = \frac{x}{\sqrt{1+x^2}}.$$

(b) (i) Suppose $A = \{a_1, \dots, a_{|A|}\}$. Then we can encode the function f by writing out the corresponding element of B as a "word", where the ith letter corresponds to $f(a_i)$:

$$\underset{a_1}{\underline{\quad}} \quad \underset{a_2}{\underline{\quad}} \quad \underset{a_3}{\underline{\quad}} \quad \cdots \quad \underset{a_{|A|}}{\underline{\quad}}$$

in each space, we can put any of the $|B|$ possible output, or 'undefined', so we have $1+|B|$ options for each a_i (since the function doesn't need to be total).

Thus the number of functions is $(1+|B|)(1+|B|)\cdots(1+|B|) = (1+|B|)^{|A|}$. □

(ii) Similar to part (i), but the function must be total, so instead of $|B|+1$, we have $|B|$ choices. Moreover, we cannot repeat ourselves, since f is injective. Thus $|B|(|B|-1)\cdots(|B|-|A|+1)$ is the result, assuming $|A| \leqslant |B|$. If $|A| > |B|$, then the answer is zero.

(c) (i) $[2,17]$ (ii) $\{82\}$ (iii) $[2,5]\cup(10,17]$

(iv) $[1,9]$ (v) $\{-\sqrt{2},\sqrt{2}\}$ (vi) $[-\sqrt{2},\sqrt{2}]$

(d) (i)
$$\begin{aligned}
x\in f^{-1}(A\cup B) &\iff f(x)\in A\cup B && \text{(definition of } f^{-1}\text{)}\\
&\iff f(x)\in A\vee f(x)\in B && \text{(definition } \cup\text{)}\\
&\iff x\in f^{-1}(A)\vee x\in f^{-1}(B) && \text{(definition of } f^{-1}\text{)}\\
&\iff x\in f^{-1}(A)\cup f^{-1}(B) && \text{(definition of } \cup\text{),}
\end{aligned}$$

which completes the proof. □

(ii) It is true, and the proof is similar:

$$\begin{aligned}
&y\in f(A\cup B)\\
&\iff \exists x:(x\in A\cup B\wedge f(x)=y) && \text{(definition of } f(S)\text{)}\\
&\iff \exists x:((x\in A\vee x\in B)\wedge f(x)=y) && \text{(definition of } \cup\text{)}\\
&\iff \exists x:(x\in A\wedge f(x)=y)\vee(x\in B\wedge f(x)=y) && \text{(distributivity)}\\
&\iff y\in f(A)\vee y\in f(B) && \text{(definition of } f(S)\text{)}\\
&\iff y\in f(A)\cup f(B), && \text{(definition of } \cup\text{)}
\end{aligned}$$

as required. □

4. (a) $\forall k>0,\exists x,y\in A:|f(x)-f(y)|>k|x-y|$.

(b) Use lc.mt/tt for this.

(c) To show that $(A\setminus B)\cup(B\setminus A)=(A\cup B)\setminus(A\cap B)$, we must prove both:

(i) $(A\setminus B)\cup(B\setminus A)\subseteq(A\cup B)\setminus(A\cap B)$

(ii) $(A\cup B)\setminus(A\cap B)\subseteq(A\setminus B)\cup(B\setminus A)$

For (i), we have

$$\begin{aligned}
&x\in(A\setminus B)\cup(B\setminus A)\\
\implies &x\in(A\setminus B)\vee x\in(B\setminus A) && \text{(definition of } \cup\text{)}\\
\implies &(x\in A\wedge x\notin B)\vee(x\in B\wedge x\notin A), && \text{(definition of } \setminus\text{)}\\
\implies &((x\in A\wedge x\notin B)\vee x\in B) && \text{(distributivity of}\\
&\quad\wedge((x\in A\wedge x\notin B)\vee x\notin A), && \quad\vee\text{ over }\wedge\text{)}\\
\implies &((x\in A\vee x\in B)\wedge(x\notin B\vee x\in B)) && \text{(distributivity of}\\
&\wedge((x\in A\vee x\notin A)\wedge(x\notin B\vee x\notin A)), && \quad\vee\text{ over }\wedge\text{)}
\end{aligned}$$

$$\begin{aligned}
&\implies ((x \in A \vee x \in B) \wedge \text{true}) \\
&\quad \wedge(\text{true} \wedge (x \notin B \vee x \notin A)), && (\phi \vee \neg\phi \leftrightarrow \text{true}) \\
&\implies (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B), && (\phi \wedge \text{true} \leftrightarrow \phi) \\
&\implies (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B), && \text{(de Morgan's)} \\
&\implies x \in (A \cup B) \wedge x \notin (A \cap B), && (\text{definition of } \cup, \cap) \\
&\implies x \in (A \cup B) \smallsetminus (A \cap B), && (\text{definition of } \smallsetminus)
\end{aligned}$$

and for (ii), notice that every step above is reversible, so replacing each $\Rightarrow$ with a $\Leftrightarrow$ gives us a complete proof. □

(d) (i) The relation is reflexive, since each month obviously starts on the same day of the week as itself.

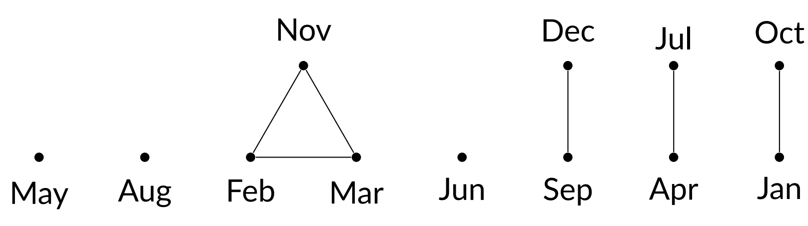
Similarly, $\sim$ is obviously symmetric since m_1 starts on the same day as m_2 is equivalent to m_2 starting on the same day as m_1.

Transitivity is similarly trivial: If m_1 and m_2 start on the same day, and m_2 and m_3 start on the same day, then clearly m_1 and m_3 start on the same day. □

(ii) For 2023, the classes are:

Mon: {May}
Tues: {Aug}
Wed: {Feb, Mar, Nov}
Thu: {Jun}
Fri: {Sep, Dec}
Sat: {Apr, Jul}
Sun: {Jan, Oct}

(iii)



(e) By Pythagoras' theorem, we must have $(m+1)^2 = a^2 + m^2$, which means that $2m+1 = a^2$, in other words, a^2 is odd. This can only happen if a is odd. □