L-Università ta' Malta

Department of Mathematics
**Faculty of Science**

**B.Sc. (Hons.) Year I**

Semester I Examination Session 2022/23

MAT1804: Mathematics for Computing — 23rd January 2023, 13:00–15:05

# Instructions

Read the following instructions carefully.

- Attempt only **THREE** questions.
- Each question carries **35** marks.
- Calculators and mathematical formulæ booklet will be provided.

⚠ Attempt only **THREE** questions.

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**Question 1.**

(a) Use a truth table to show that $(\varphi \wedge (\neg\varphi \vee \psi)) \vee \psi \leftrightarrow \psi$ is a tautology.

(b) A function $f: A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ is said to be *continuous* if

$$\forall a \in A, \forall \epsilon > 0, \exists \delta > 0 : \forall x \in A, \ (0 < |x-a| < \delta \implies |f(x) - f(a)| < \epsilon).$$

Write out, in symbols, the negation of this statement.

(c) Write out the elements of the following sets.

(i) $(1,8] \cap 2\mathbb{Z}$ (ii) $\{x \in \mathbb{R} : x^2 < 6x + 7\} \cap \mathbb{Z}$

(d) Let $A, B, C \subseteq \Omega$, and for all subsets $S \subseteq \Omega$, let $\overline{S} := \Omega \setminus S$. Prove that:

(i) If $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$ (ii) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

(iii) $A \setminus B = A \cap \overline{B}$ (iv) $\overline{A \setminus B} = \overline{A} \cup B$

**[5, 5, 5, 20 marks]**

**Question 2.**

(a) You may assume that $n \in \mathbb{Z}$ is not a multiple of 3 if and only if it can be expressed in the form $3k+1$ or $3k+2$ for an appropriate $k \in \mathbb{Z}$.

(i) Show that if $n^3$ is divisible by 3, then $n$ is.

(ii) Hence, show that $\sqrt[3]{3}$ is irrational.

(iii) Show, by contradiction, that $\dfrac{1+2\sqrt[3]{3}}{\sqrt[3]{3}-1}$ is irrational.

(b) (i) Show that $k^2 + k$ is even for any $k \in \mathbb{Z}$.

(ii) If $a$ is odd, show that the equation $8x + a^2 = 1$ always has an integer solution $x \in \mathbb{Z}$.

(c) Using induction, show that

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{3}n(n+\tfrac{1}{2})(n+1).$$

**[15, 10, 10 marks]**

**Question 3.**

(a) *Qagħaq tal-għasel* are sold in packets of 3 or 10. Show that you can purchase any number of *qagħaq* greater than 17.

[*In other words, show that any whole number* $n > 17$ *can be expressed as a sum of 3's and 10's only.*]

(b) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (5x + 2y, 2x + y)$ is a bijection, and find a formula for the inverse $f^{-1}$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. Find:

(i) $f([1,4])$ (ii) $f(\{9\})$ (iii) $f([1,2] \cup (3,4])$

(iv) $f([-3,3])$ (v) $f^{-1}(\{3\})$ (vi) $f^{-1}([-3,3])$

(d) Let $f : X \to Y$ be any function.

(i) Show that for any $A, B \subseteq Y$, we have $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(ii) Is it always true that $f(A \cup B) = f(A) \cup f(B)$ for $A, B \subseteq X$? If yes, prove it, if no, give a counterexample.

**[10, 10, 6, 9 marks]**

**Question 4.**

(a) (i) State and prove the handshaking lemma.

(ii) Show that $\delta(G) \leqslant (n-1)\rho(G) \leqslant \Delta(G)$, where $\rho(G)$ denotes the *density* of the graph $G$, and $n = |V(G)|$.

(b) Show that the degrees of a graph cannot all be distinct.
[*Hint: use the pigeonhole principle*.]

(c) Let $G$ be a connected graph, and let $P$ be a *shortest* path joining the vertices $x$ and $y$ in $G$. Show that any vertex $v \in V(G)$ cannot have more than 3 neighbours on the path $P$.

(d) (i) Show that any tree has at least two leaves.

(ii) Prove that the number of edges in a tree on $n$ vertices is $n-1$.

(iii) Show that a tree on $n$ vertices whose degrees are all either 1 or 3 has precisely $\frac{n}{2} + 1$ leaves.
[*Hint: use the handshaking lemma*.]

**[10, 7, 6, 12 marks]**

# Answers and Hints

1. (a) Use https://lc.mt/tt to check this.

   (b) $\exists a \in A, \exists \epsilon > 0 : \forall \delta > 0, \exists x \in A : 0 < |x - a| < \delta \wedge |f(x) - f(a)| \geqslant \epsilon$.

   (c) (i) $\{2, 4, 6, 8\}$ (ii) $\{0, 1, 2, 3, 4, 5, 6\}$

   (d) (i)
$$\begin{aligned} x \in \overline{B} &\implies x \in \Omega \smallsetminus B && \text{(definition of } \bar{S}\text{)} \\ &\implies x \in \Omega \wedge x \notin B && \text{(definition of } \smallsetminus\text{)} \\ &\implies x \in \Omega \wedge x \notin A && \text{(contrapositive of definition of } A \subseteq B\text{)} \\ &\implies x \in \Omega \smallsetminus A && \text{(definition of } \smallsetminus\text{)} \\ &\implies x \in \overline{A} && \text{(definition of } \bar{S}\text{)} \end{aligned}$$

   Thus $\overline{B} \subseteq \overline{A}$. □

   (ii)
$$\begin{aligned} x \in \overline{A \cup B} &\iff x \in \Omega \smallsetminus (A \cup B) && \text{(definition of } \bar{S}\text{)} \\ &\iff x \in \Omega \wedge x \notin A \cup B && \text{(definition of } \smallsetminus\text{)} \\ &\iff x \in \Omega \wedge \neg(x \in A \cup B) && \text{(definition of } \notin\text{)} \\ &\iff x \in \Omega \wedge \neg(x \in A \vee x \in B) && \text{(definition of } \cup\text{)} \\ &\iff x \in \Omega \wedge (x \notin A \wedge x \notin B) && \text{(de Morgan's law)} \\ &\iff x \in \Omega \wedge x \in \Omega \wedge (x \notin A \wedge x \notin B) && (\varphi \leftrightarrow \varphi \wedge \varphi) \\ &\iff (x \in \Omega \wedge x \notin A) \wedge (x \in \Omega \wedge x \notin B) \\ & && \text{(associativity \& commutativity)} \\ &\iff (x \in \Omega \smallsetminus A) \wedge (x \in \Omega \smallsetminus B) && \text{(definition of } \smallsetminus\text{)} \\ &\iff x \in \overline{A} \wedge x \in \overline{B} && \text{(definition of } \bar{S}\text{)} \\ &\iff x \in \overline{A} \cap \overline{B}, && \text{(definition of } \cap\text{)} \end{aligned}$$

   Therefore $\overline{A \cup B} = \overline{A} \cap \overline{B}$. □

   (iii)
$$\begin{aligned} x \in A \smallsetminus B &\iff x \in A \wedge x \notin B && \text{(definition of } \smallsetminus\text{)} \\ &\iff x \in A \wedge x \in A \wedge x \notin B && (\varphi \leftrightarrow \varphi \wedge \varphi) \\ &\iff x \in A \wedge x \in \Omega \wedge x \notin B && (A \subseteq \Omega) \\ &\iff x \in A \wedge (x \in \Omega \smallsetminus B) && \text{(definition of } \smallsetminus\text{)} \\ &\iff x \in A \wedge (x \in \overline{B}) && \text{(definition of } \overline{S}\text{)} \\ &\iff x \in A \cap \overline{B}, && \text{(definition of } \cap\text{)} \end{aligned}$$

   Thus $A \smallsetminus B = A \cap \overline{B}$. □

(iv) $$\begin{aligned} x \in \overline{A \smallsetminus B} &\iff x \in \Omega \smallsetminus (A \smallsetminus B) && \text{(definition of } \overline{S}\text{)} \\ &\iff x \in \Omega \wedge x \notin (A \smallsetminus B) && \text{(definition of } \smallsetminus\text{)} \\ &\iff x \in \Omega \wedge \neg(x \in A \wedge x \notin B) && \text{(definition of } \smallsetminus\text{)} \\ &\iff x \in \Omega \wedge (x \notin A \vee x \in B) && \text{(de Morgan's law)} \\ &\iff (x \in \Omega \wedge x \notin A) \vee (x \in \Omega \wedge x \in B) && \text{(distributivity)} \\ &\iff (x \in \Omega \wedge x \notin A) \vee x \in B && (\wedge\text{-elimination}/B \subseteq \Omega) \\ &\iff (x \in \Omega \smallsetminus A) \vee x \in B && \text{(definition of } \smallsetminus\text{)} \\ &\iff x \in \overline{A} \vee x \in B && \text{(definition of } \overline{S}\text{)} \\ &\iff x \in \overline{A} \cup B, && \text{(definition of } \cup\text{)} \end{aligned}$$

Thus $\overline{A \smallsetminus B} = \overline{A} \cup B$. □

2. (a) (i) By contrapositive: we show that if $n$ is not divisible by 3, then $n^3$ is not divisible by 3.

Indeed, if $n$ is not divisible by 3, then it equals $3k+1$ or $3k+2$ for appropriate $k$. In the first case,

$$n^3 = (3k+1)^3 = 3(9k^3 + 9k^2 + 3k) + 1,$$

so it is not divisible by 3. In the second case,

$$n^3 = (3k+2)^3 = 3(9k^3 + 18k^2 + 12 + 2) + 2,$$

so it is also not divisible by 3. □

(ii) By contradiction: suppose that we can write $\sqrt[3]{3} = a/b$ with $a, b \in \mathbb{Z}$ and $\operatorname{hcf}(a,b) = 1$. Then $a^3 = 3b^3$, so $a^3$ is a multiple of 3, which by (i) implies that $a$ is a multiple of 3, say, $a = 3k$. But then $(3k)^3 = 3b^3$ which implies that $b^3 = 3(3k^3)$, so $b^3$ is also a multiple of 3, which again by (i) implies that $b$ is a multiple of 3. This contradicts that $\operatorname{hcf}(a,b) = 1$. □

(iii) By contradiction: If the given number is rational, say equal to $a/b$, then we may express

$$\sqrt[3]{3} = \frac{a+b}{a-2b},$$

which contradicts (ii). □

(b) (i) $k^2+k=k(k+1)$ is the product of two consecutive numbers, so one of them must be even. Thus the product is even.

(ii) If $a$ is odd, then $a=2k+1$. Thus the equation is $8x+(2k+1)^2=1$, which expands to $8x+4(k^2+k)=0$. Since $k^2+k$ is even, say equal to $2b$, then the equation is $8x+8b=0$, which clearly has the solution $x=-b$.

(c) When $n=1$, LHS $=1^2=1$ and RHS $=\frac{1}{3}(1)(1+\frac{1}{2})(1+1)=1$, so the result holds. Now for the inductive step,

$$\begin{aligned} 1^2+2^2+\cdots+n^2 &= \tfrac{1}{3}(n-1)(n-\tfrac{1}{2})n+n^2 && \text{(by IH)} \\ &= \tfrac{1}{3}\,n\left[(n-1)(n-\tfrac{1}{2})+3n\right] \\ &= \tfrac{1}{3}\,n\left[n^2+\tfrac{3}{2}n+\tfrac{1}{2}\right] \\ &= \tfrac{1}{3}\,n(n+\tfrac{1}{2})(n+1), \end{aligned}$$

which completes the proof. □

3. (a) When $n=18$, we can write $18=3+3+3+3+3+3$ which starts the induction. Now for any $n>18$, suppose we have expressed (by the IH) $n-1$ as a sum of 3's and 10's. If there are at least three 3's in this representation, then replacing any three of them with a single 10 transforms the representation of $n-1$ into one of $n$.

Now for the remaining case, suppose that there are at most two 3's in the representation. Then there must be at least two 10's (otherwise the number would be too small). Replacing any two 10s with seven 3's will transform the representation of $n-1$ into one of $n$. □

(b) We need to show that $f$ is (i) functional, (ii) total, (iii) injective and (iv) surjective.

(i) Clearly $f$ is functional, since it unambiguously assigns a unique pair of coordinates to each input pair $(x,y)\in\mathbb{R}^2$.

(ii) It is also clear that $f$ is total, since it assigns every point in the domain $\mathbb{R}^2$ a corresponding pair of coordinates.

(iii) To see that $f$ is injective, suppose that $f(x,y) = f(a,b)$, i.e.,

$$(5x+2y, 2x+y) = (5a+2b, 2a+b)$$

$$\implies \begin{cases} 5x+2y = 5a+2b & (1) \\ 2x+\ \ y = 2a+b & (2) \end{cases}$$

$$\implies \begin{cases} 5x+2y = 5a+2b & (1) \\ 4x+2y = 4a+2b & 2\cdot(2) \end{cases}$$

and subtracting the two equations gives us that $x = a$, and then by (2) we clearly get that $y = b$. Thus, if $f(x,y) = f(a,b)$, then $(x,y) = (a,b)$, so that $f$ is injective.

(iv) Finally, to see that $f$ is surjective, take any point $(x,y)$ in the codomain $\mathbb{R}^2$, and solve

$$\begin{aligned} & f(a,b) = (x,y) \\ \implies\quad & (5a+2b, 2a+b) = (x,y) \\ \implies\quad & \begin{cases} 5a+2b = x & (1) \\ 2a+\ \ b = y & (2) \end{cases} \\ \implies\quad & \begin{cases} 5a+2b = x & (1) \\ 4a+2b = 2y & 2\cdot(2) \end{cases} \end{aligned}$$

subtracting gives $a = x-2y$, and then using (2), we get that $b = y - 2(x-2y) = 5y-2x$. Thus we see that

$$f(x-2y, 5y-2x) = (x,y),$$

and since this works for all $(x,y)$ in the codomain, we see that $f$ is surjective.

We also immediately obtain a formula for $f^{-1}$, namely,

$$f^{-1}(x,y) = (x-2y, 5y-2x).$$

(c) (i) $[1,2]$ (ii) $\{3\}$ (iii) $[1,\sqrt{2}] \cup (\sqrt{3},2]$

(iv) $[0,\sqrt{3}]$ (v) $\{9\}$ (vi) $[0,9]$

(d) (i) $$\begin{aligned} x \in f^{-1}(A \cup B) &\iff f(x) \in A \cup B && \text{(definition of } f^{-1}\text{)} \\ &\iff f(x) \in A \vee f(x) \in B && \text{(definition } \cup\text{)} \\ &\iff x \in f^{-1}(A) \vee x \in f^{-1}(B) && \text{(definition of } f^{-1}\text{)} \\ &\iff x \in f^{-1}(A) \cup f^{-1}(B) && \text{(definition of } \cup\text{),} \end{aligned}$$

which completes the proof. □

(ii) It is true, and the proof is similar:

$$\begin{aligned} & y \in f(A \cup B) \\ \iff & \exists x : (x \in A \cup B \wedge f(x) = y) && \text{(definition of } f(S)\text{)} \\ \iff & \exists x : ((x \in A \vee x \in B) \wedge f(x) = y) && \text{(definition of } \cup\text{)} \\ \iff & \exists x : (x \in A \wedge f(x) = y) \vee (x \in B \wedge f(x) = y) && \text{(distributivity)} \\ \iff & y \in f(A) \vee y \in f(B) && \text{(definition of } f(S)\text{)} \\ \iff & y \in f(A) \cup f(B), && \text{(definition of } \cup\text{)} \end{aligned}$$

as required. □

4. (a) (i) For all graphs $G$,

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

*Proof 1.* The number $\deg(v)$ counts the number of edges incident to the vertex $v$. Since each edge in the graph is incident to precisely two vertices, then each edge in $|E(G)|$ contributes 2 to the sum. □

*Proof 2.* We have

$$\begin{aligned} \sum_{v \in V(G)} \deg(v) &= \sum_{v \in V(G)} |N(v)| \\ &= \sum_{v \in V(G)} \sum_{e \in E(G)} \mathbb{1}_{v \in e} \\ &= \sum_{e \in E(G)} \sum_{v \in V(G)} \mathbb{1}_{v \in e} = \sum_{e \in E(G)} 2 = 2|E(G)|. \end{aligned}$$ □

(ii) First of all, observe that

$$(n-1)\rho(G) = \frac{(n-1)|E(G)|}{n(n-1)/2} = \frac{2|E(G)|}{|V(G)|}.$$

Now clearly

$$\sum_{v\in V(G)} \delta(G) \leqslant \sum_{v\in V(G)} \deg(v) \leqslant \sum_{v\in V(G)} \Delta(G)$$
$$\implies |V(G)|\delta(G) \leqslant 2|E(G)| \leqslant |V(G)|\Delta(G),$$

and dividing through by $|V(G)|$ gives

$$\delta(G) \leqslant \frac{2|V(G)|}{|E(G)|} \leqslant \Delta(G),$$

where the middle term is $(n-1)\rho(G)$ as observed earlier. □

(b) Consider first the case of graphs which have no vertex of degree zero. If the number of vertices is $n$, then every vertex must satisfy $1 \leqslant \deg(v) \leqslant n-1$, so there are $n-1$ possible degrees which we must assign to $n$ vertices, and so the pigeonhole principle ensures that there must be at least two vertices which receive the same degree.

Now consider graphs which do contain a vertex of degree zero. If there is more than one such vertex, then we are done, so suppose there is precisely one vertex of degree zero. If we temporarily ignore it, then we get a graph on $n-1$ vertices which has no vertex of degree zero, and we can apply the argument from earlier since now we must have $1 \leqslant \deg(v) \leqslant n-2$ for all of the $n-1$ vertices, and so the pigeonhole principle guarantees that a pair of vertices in this subgraph receive the same degree. □

(c) By contradiction: if $v$ has neighbours $u_1, u_2, u_3, u_4$ on the shortest path from $x$ to $y$ (suppose these are labelled according to the order one encounters them when travelling along the path), then going from $x$ to $v_1$ and then to $v_4$ is a "shortcut", contradicting that $P$ is the shortest path from $x$ to $y$. □

(d) (i) Let $P$ be a longest path in the tree. This necessarily has two leaves at its end, since otherwise it is not a longest path.

(ii) By induction on $n$. Clearly when $n = 1$ we have $0 = n-1$ edges, which establishes the base case. Now given a tree $T$ on $n$ vertices, remove a leaf $\ell$ (guaranteed to exist by (i)) to get $T-\ell$, which by the IH has $(n-1)-1 = n-2$ edges. But adding $\ell$ back increases the number of edges by 1, so we have $n-1$ edges. □

(iii) Suppose there are $k$ vertices of degree 1. Then there are $n-k$ vertices of degree 3, and so the sum of degrees is $k+3(n-k)$, which by the handshaking lemma is $2|E(G)| = 2(n-1)$. Solving the equation $k+3(n-k) = 2(n-1)$ for $k$ gives $k = \frac{n}{2}+1$. $\square$