L-Università ta' Malta

Department of Mathematics
Faculty of Science

B.Sc. (Hons.) Year I

Summer Examination Session 2024

MAT1804: Mathematics for Computing

4th September 2024
08:30–10:35

Instructions

Read the following instructions carefully.

- Attempt only **THREE** questions.
- Each question carries **35** marks.
- Calculators and mathematical formulæ booklet will be provided.

⚠ Attempt only **THREE** questions.

Question 1.

(a) Construct a truth table for the proposition $(\varphi \wedge \psi) \vee \xi \to \varphi \vee \psi \vee \xi$.

(b) A total function $f : \mathbb{R} \to \mathbb{R}$ is said to be differentiable if

$$\exists D \in \mathbb{R} : \forall x, h \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0 : |h| < \delta \to \left| f(x+h) - f(x) - Dh \right| < \epsilon.$$

Write down the negation of this statement.

(c) Prove the following for any sets A, B, C:

(i) $A \cap (A \cup B) = A$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(d) For any $x, y \in \mathbb{Z}$, write $x \sim y$ if $3x + 7y$ is even.

(i) Show that $\sim$ is an equivalence relation.

(ii) What is the equivalence class of the number 5 under this relation? Write down the equivalence class of x for $x \in \mathbb{Z}$ using set notation.

[5, 5, 15, 10 marks]

Question 2.

(a) Consider the matrix $\mathbf{A} = \begin{pmatrix} 4 & 2 & 4 \\ 1 & -3 & 2 \\ -2 & -2 & -2 \end{pmatrix}$.

(i) Given that $\mathbf{A}$ has eigenvalues $-2, -1$ and 2, find a corresponding eigenvector for each eigenvalue.

(ii) Work out $\mathbf{A}^2$ and $\mathbf{A}^3$. Hence, verify that $\mathbf{A}^3 + \mathbf{A}^2 = 4\mathbf{A} + 4\mathbf{I}$, and use this to determine $\mathbf{A}^{-1}$.

[Hint: multiply throughout the equation by $\mathbf{A}^{-1}$.]

(iii) Using $\mathbf{A}^{-1}$, solve the following system of simultaneous equations:

$$\begin{cases} 4x + 2y + 4z = 4 \\ x - 3y + 2z = 9 \\ -2x - 2y - 2z = 0. \end{cases}$$

(b) Let G be a graph with n vertices and m edges.

(i) Show that

$$\sum_{v \in V(G)} \deg(v) = 2m.$$

(ii) Hence, show that $n \cdot \rho(G) \geqslant \delta(G)$, where $\rho(G)$ and $\delta(G)$ denote the density and the minimum degree of G respectively.

(c) (i) Show that any tree has at least two leaves.

(ii) Prove that the number of edges in a tree on n vertices is $n-1$.

(iii) Show that a tree on n vertices whose degrees are all either 1 or 3 has precisely $\frac{n}{2}+1$ leaves.
[*Hint: use the handshaking lemma*.]

[15, 8, 12 marks]

Question 3.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + \frac{1}{x}$.

(a) Find:

(i) $f([1,4])$

(ii) $f(\{9\})$

(iii) $f([1,2] \cup (3,4])$

(iv) $f^{-1}(\{4\})$

(b) Let $g = f \restriction [1,\infty)$. Prove that g is an injection, and find a formula for g^{-1}.

(c) Show that the sum of a positive rational number and its reciprocal is always greater than or equal to 2.

(d) Let $f : X \to Y$ be a function, and let $A, B \subseteq Y$.

(i) Show that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(ii) Show that $f(f^{-1}(A)) \subseteq A$.

(e) Using induction, prove that

$$\frac{1}{\sqrt{0}+\sqrt{1}} + \frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1}.$$

[4, 6, 8, 8, 9 marks]

Question 4.

(a) You may assume that if an integer n is not divisible by 3 then it must equal $3k+1$ or $3k+2$ for some integer k.

(i) Show that if x^5 is divisible by 3, then x is divisible by 3.

(ii) Hence, show that $\sqrt[5]{3}$ is irrational.

(iii) Deduce that $\dfrac{1+\sqrt[5]{3}}{1-\sqrt[5]{3}}$ is irrational.

(b) Prove that the area of an equilateral triangle with base b is $\frac{\sqrt{3}}{4}b^2$.

(c) Let A and B be two finite sets.

(i) Show that the number of functions from A to B is $(|B|+1)^{|A|}$.

[*Recall that for us, a function need not be total.*]

(ii) How many total injective functions are there from A to B?

(d) Use strong induction to prove that every integer $n \geqslant 2$ can be written as a product of prime numbers.

[12, 6, 10, 7 marks]

Answers and Hints

1. (a) Use `lc.mt/tt` for this.

   (b) $\forall D \in \mathbb{R}, \exists x, h \in \mathbb{R}, \exists \epsilon > 0 : \forall \delta > 0, |h| < \delta \wedge |f(x+h) - f(x) - Dh| \geqslant \epsilon$.

   (c) Straightforward set theory proofs.

   (d) (i) Hint: For reflexivity, $3x + 7x = 10x$, which is even. For symmetry, $3y + 7x = (10x + 10y) - (3x + 7y)$, which is even. Finally for reflexivity, $3x + 7z = (3x + 7y) + (3y + 7z) - 10y$ which is even.

   (ii) $[5] = \{y : y \text{ is odd}\}$, and $[x] = \{y : y \text{ has the same parity as } x\}$.

2. (a) (i) For -2: $(-1, 1, 1)$, for -1: $(-2, 1, 2)$, for 2: $(-2, 0, 1)$.

   (ii) $\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 5 & -2 & 8 \\ -1 & 0 & -2 \\ -4 & 2 & -7 \end{pmatrix}$.

   (iii) $\mathbf{A}^{-1}(4, 9, 0) = (1, -2, 1)$, so $x = 1 = z, y = -2$.

   (b) (i) *Proof 1.* The number $\deg(v)$ counts the number of edges incident to the vertex v. Since each edge in the graph is incident to precisely two vertices, then each edge in $|E(G)|$ contributes 2 to the sum. □

   Proof 2. We have

$$\begin{aligned}\sum_{v \in V(G)} \deg(v) &= \sum_{v \in V(G)} |N(v)| \\ &= \sum_{v \in V(G)} \sum_{e \in E(G)} \mathbb{1}_{v \in e} \\ &= \sum_{e \in E(G)} \sum_{v \in V(G)} \mathbb{1}_{v \in e} = \sum_{e \in E(G)} 2 = 2|E(G)|. \end{aligned}$$ □

   (ii) First of all, observe that

$$(n-1)\rho(G) = \frac{(n-1)|E(G)|}{n(n-1)/2} = \frac{2|E(G)|}{|V(G)|}.$$

Now clearly

$$\sum_{v \in V(G)} \delta(G) \leqslant \sum_{v \in V(G)} \deg(v)$$
$$\implies |V(G)|\delta(G) \leqslant 2|E(G)|,$$

and dividing through by $|V(G)|$ gives

$$\delta(G) \leqslant \frac{2|V(G)|}{|E(G)|} = (n-1)\rho(G) \leqslant n \cdot \rho(G),$$

as required. □

(c) (i) Let P be a longest path in the tree. This necessarily has two leaves at its end, since otherwise it is not a longest path.

(ii) By induction on n. Clearly when $n = 1$ we have $0 = n-1$ edges, which establishes the base case. Now given a tree T on n vertices, remove a leaf ℓ (guaranteed to exist by (i)) to get $T - \ell$, which by the IH has $(n-1)-1 = n-2$ edges. But adding ℓ back increases the number of edges by 1, so we have $n-1$ edges. □

(iii) Suppose there are k vertices of degree 1. Then there are $n-k$ vertices of degree 3, and so the sum of degrees is $k+3(n-k)$, which by the handshaking lemma is $2|E(G)| = 2(n-1)$. Solving the equation $k+3(n-k) = 2(n-1)$ for k gives $k = \frac{n}{2}+1$. □

3. (a) (i) $[2, \frac{17}{4}]$

(ii) $\{\frac{82}{9}\}$

(iii) $[2, \frac{5}{2}] \cup (\frac{10}{3}, \frac{17}{4}]$

(iv) $\{2-\sqrt{3}, 2+\sqrt{3}\}$

(b) $g^{-1}(x) = \frac{1}{2}(x + \sqrt{x^2-4})$.

(c) Hint: rewrite $\frac{x}{y} + \frac{y}{x} = \frac{(x+y)^2}{xy} - 2$

(d) Straightforward set proofs.

(e) Straightforward induction proof.

4. (a) (i) By contrapositive, show that if x is not divisible by 3, then neither is x^5. Use the assumption and consider the two cases separately.

(ii) Identical to the proof for $\sqrt{2}$ in the notes, using (i) instead of the appropriate lemma.

(iii) By contradiction: if this were rational, say equal to a/b, then we could solve for $\sqrt[5]{3}$ and write $\sqrt[5]{3} = \frac{a-b}{a+b}$, which would be rational, contradicting (ii).

(b) Proof outline: Construct a diagram of an equilateral triangle, and split it into two right-angled triangles with hypotenuse b and base $b/2$. Consequently, we can use Pythagoras' theorem to find the height: $b^2 = (b/2)^2 + h^2$, and then using the usual formula $A = bh/2$, we get the result.

(c) (i) Suppose $A = \{a_1, \dots, a_{|A|}\}$. Then we can encode the function f by writing out the corresponding element of B as a "word", where the ith letter corresponds to $f(a_i)$:

$$\underset{a_1}{\underline{\quad}} \quad \underset{a_2}{\underline{\quad}} \quad \underset{a_3}{\underline{\quad}} \quad \cdots \quad \underset{a_{|A|}}{\underline{\quad}}$$

in each space, we can put any of the $|B|$ possible output, or 'undefined', so we have $1 + |B|$ options for each a_i (since the function doesn't need to be total).

Thus the number of functions is $(1+|B|)(1+|B|)\cdots(1+|B|) = (1+|B|)^{|A|}$. □

(ii) Similar to part (i), but the function must be total, so instead of $|B|+1$, we have $|B|$ choices. Moreover, we cannot repeat ourselves, since f is injective. Thus $|B|(|B|-1)\cdots(|B|-|A|+1)$ is the result, assuming $|A| \leqslant |B|$. If $|A| > |B|$, then the answer is zero.

(d) Standard from notes.