L-Università ta' Malta

Department of Mathematics
Faculty of Science

B.Sc. (Hons.) Year I

Semester I Examination Session 2023/24

MAT1804: Mathematics for Computing — 24th January 2024, 8:30–10:35

Instructions

Read the following instructions carefully.

- Attempt only **THREE** questions.
- Each question carries **35** marks.
- Calculators and mathematical formulæ booklet will be provided.

⚠ Attempt only **THREE** questions.

Question 1.

(a) (i) State and prove the handshaking lemma.

(ii) Show that

$$\delta(G) \leqslant (n-1)\rho(G) \leqslant \Delta(G),$$

where $\delta(G)$, $\rho(G)$ and $\Delta(G)$ denote the *minimum degree*, *density* and *maximum degree* of the graph G respectively, and $n = |V(G)|$.

(b) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -5 & 1 & 8 \\ 2 & 0 & -4 \\ -4 & 1 & 7 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} -4 & -1 & 4 \\ -2 & 3 & 4 \\ -2 & -1 & 2 \end{pmatrix}.$$

(i) Work out the product **AB**, and deduce $\mathbf{A}^{-1}$.

(ii) Hence, solve the system of equations

$$\begin{cases} -5x + y + 8z = 9 \\ 2x - 4z = -2 \\ -4x + y + 7z = 7. \end{cases}$$

(iii) Given that

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda+5 & -1 & -8 \\ -2 & \lambda & 4 \\ 4 & -1 & \lambda-7 \end{vmatrix} = (\lambda+1)(\lambda-1)(\lambda-2),$$

find the eigenvalues of **A**, and determine an eigenvector for each.

(c) By constructing an appropriate diagram involving the basis vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, find the matrix **R** which rotates vectors in $\mathbb{R}^2$ by 45° anticlockwise about the origin.

(d) Let **A** and **B** be any two $n \times n$ matrices. Show that if $\mathbf{AP} = \mathbf{PB}$ for some invertible matrix **P**, then **A** and **B** have the same eigenvalues.

[10, 14, 5, 6 marks]

Question 2.

(a) A total function $f: A \to \mathbb{R}$ is said to be *Lipschitz continuous* if

$$\exists k > 0 : \forall x, y \in A, |f(x) - f(y)| \leqslant k|x - y|.$$

Write down the negation of the above statement.

(b) Prove that for any two sets A and B,

$$(A \smallsetminus B) \cup (B \smallsetminus A) = (A \cup B) \smallsetminus (A \cap B).$$

(c) Construct a truth table for the proposition $(\neg\varphi \wedge \psi) \vee \xi \to \neg\psi \vee \neg\xi$.

(d) For any $x, y \in \mathbb{R}$, write $x \sim y$ if $x^2 - y^2$ is an integer.

(i) Show that $\sim$ is an equivalence relation.

(ii) What is the equivalence class of the real number 2 in this relation? Write down the equivalence class of x for $x \in \mathbb{R}$ using set notation.

[5, 15, 5, 10 marks]

Question 3.

(a) If $x, y > 0$, show that $\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2 \geqslant 2$.

(b) Using induction, prove that

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \tfrac{1}{6}n(n+1)(2n+7).$$

(c) You may assume that $n \in \mathbb{Z}$ is not a multiple of 3 if and only if it can be expressed in the form $3k+1$ or $3k+2$ for an appropriate $k \in \mathbb{Z}$.

(i) Show that if n^4 is divisible by 3, then n is.

(ii) Hence, show that $\sqrt[4]{3}$ is irrational.

(iii) Show, by contradiction, that $\dfrac{1+\sqrt[4]{3}}{\sqrt[4]{3}-2}$ is irrational.

[10, 10, 15 marks]

Question 4.

(a) Let A and B be two finite sets.

(i) Show that the number of functions from A to B is $(|B|+1)^{|A|}$.

[*Recall that for us, a function need not be total.*]

(ii) How many total injective functions are there from A to B?

(b) Let $f : \mathbb{R} \rightharpoonup \mathbb{R}$ be defined by $f(x) = x^2 - 1$. Find:

(i) $f([1,4])$ (ii) $f(\{9\})$ (iii) $f([1,2] \cup (3,4])$

(iv) $f([-3,3])$ (v) $f^{-1}(\{3\})$ (vi) $f^{-1}([-3,3])$

(c) Define the function $f : (-1,1) \rightharpoonup \mathbb{R}$ by

$$f(x) = \frac{x}{\sqrt{1-x^2}}.$$

Prove that f is a bijection, hence determine a formula for f^{-1}.

(d) Let $f : X \rightharpoonup Y$ be any function.

(i) Show that for any $A, B \subseteq Y$, we have $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(ii) Is it always true that $f(A \cup B) = f(A) \cup f(B)$ for $A, B \subseteq X$? If yes, prove it, if no, give a counterexample.

[10, 10, 6, 9 marks]

Answers and Hints

1. (a) (i) For all graphs G,

$$\sum_{v\in V(G)} \deg(v) = 2|E(G)|.$$

Proof 1. The number $\deg(v)$ counts the number of edges incident to the vertex v. Since each edge in the graph is incident to precisely two vertices, then each edge in $|E(G)|$ contributes 2 to the sum. □

Proof 2. We have

$$\begin{aligned}\sum_{v\in V(G)} \deg(v) &= \sum_{v\in V(G)} |N(v)| \\ &= \sum_{v\in V(G)} \sum_{e\in E(G)} \mathbb{1}_{v\in e} \\ &= \sum_{e\in E(G)} \sum_{v\in V(G)} \mathbb{1}_{v\in e} = \sum_{e\in E(G)} 2 = 2|E(G)|.\end{aligned}$$ □

(ii) First of all, observe that

$$(n-1)\rho(G) = \frac{(n-1)|E(G)|}{n(n-1)/2} = \frac{2|E(G)|}{|V(G)|}.$$

Now clearly

$$\begin{aligned}&\sum_{v\in V(G)} \delta(G) \leqslant \sum_{v\in V(G)} \deg(v) \leqslant \sum_{v\in V(G)} \Delta(G) \\ &\implies |V(G)|\delta(G) \leqslant 2|E(G)| \leqslant |V(G)|\Delta(G),\end{aligned}$$

and dividing through by $|V(G)|$ gives

$$\delta(G) \leqslant \frac{2|V(G)|}{|E(G)|} \leqslant \Delta(G),$$

where the middle term is $(n-1)\rho(G)$ as observed earlier. □

(b) (i) $\mathbf{AB} = 2\mathbf{I}$, so $\mathbf{A}^{-1} = \frac{1}{2}\mathbf{B}$.

(ii) $\mathbf{A}^{-1}(9,-2,7) = (-3,2,-1)$, so $x = -3, y = 2, z = -1$.

(iii) Eigenvalues are $-1, 1, 2$, with corresponding eigenvectors being (multiples of) $(2,0,1)$, $(1,-2,1)$, $(1,-1,1)$.

(c) Diagram should involve rotated vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, with some calculations showing working to find the "new" x- and y-coordinates.

$$\mathbf{R} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

(d) If λ is an eigenvalue of $\mathbf{A}$, i.e., $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ for some vector $\boldsymbol{x} \neq \mathbf{0}$, then $\mathbf{PBP}^{-1}\boldsymbol{x} = \lambda\boldsymbol{x}$, i.e., $\mathbf{B}(\mathbf{P}^{-1}\boldsymbol{x}) = \lambda(\mathbf{P}^{-1}\boldsymbol{x})$. Thus λ is an eigenvalue for $\mathbf{B}$ with corresponding eigenvector $\mathbf{P}^{-1}\boldsymbol{x}$. Analogous reasoning shows that any eigenvalue of $\mathbf{B}$ is also one of $\mathbf{A}$. □

2. (a) $\forall k > 0, \exists x, y \in A : |f(x) - f(y)| > k|x - y|$.

(b) To show that $(A \smallsetminus B) \cup (B \smallsetminus A) = (A \cup B) \smallsetminus (A \cap B)$, we must prove both:

(i) $(A \smallsetminus B) \cup (B \smallsetminus A) \subseteq (A \cup B) \smallsetminus (A \cap B)$

(ii) $(A \cup B) \smallsetminus (A \cap B) \subseteq (A \smallsetminus B) \cup (B \smallsetminus A)$

For (i), we have

$$\begin{aligned}
& x \in (A \smallsetminus B) \cup (B \smallsetminus A) & \\
\Longrightarrow\ & x \in (A \smallsetminus B) \vee x \in (B \smallsetminus A) & \text{(definition of } \cup) \\
\Longrightarrow\ & (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A), & \text{(definition of } \smallsetminus) \\
\Longrightarrow\ & ((x \in A \wedge x \notin B) \vee x \in B) & \text{(distributivity of} \\
& \quad \wedge((x \in A \wedge x \notin B) \vee x \notin A), & \vee \text{ over } \wedge) \\
\Longrightarrow\ & ((x \in A \vee x \in B) \wedge (x \notin B \vee x \in B)) & \text{(distributivity of} \\
& \wedge((x \in A \vee x \notin A) \wedge (x \notin B \vee x \notin A)), & \vee \text{ over } \wedge) \\
\Longrightarrow\ & ((x \in A \vee x \in B) \wedge \text{true}) & \\
& \wedge(\text{true} \wedge (x \notin B \vee x \notin A)), & (\phi \vee \neg\phi \leftrightarrow \text{true}) \\
\Longrightarrow\ & (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B), & (\phi \wedge \text{true} \leftrightarrow \phi) \\
\Longrightarrow\ & (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B), & \text{(de Morgan's)} \\
\Longrightarrow\ & x \in (A \cup B) \wedge x \notin (A \cap B), & \text{(definition of } \cup, \cap) \\
\Longrightarrow\ & x \in (A \cup B) \smallsetminus (A \cap B), & \text{(definition of } \smallsetminus)
\end{aligned}$$

and for (ii), notice that every step above is reversible, so replacing each $\Rightarrow$ with a $\Leftrightarrow$ gives us a complete proof. $\square$

(c) Use `lc.mt/tt` for this.

(d) (i) The proof is straightforward. For reflexivity, $x^2 - x^2 = 0 \in \mathbb{Z}$, so $x \sim x$ for any $x \in \mathbb{R}$.

For symmetry, $x \sim y$ means $x^2 - y^2$ is an integer, so $y^2 - x^2 = -(x^2 - y^2)$ is clearly also an integer, which means $y \sim x$.

Finally for transitivity, $x \sim y$ and $y \sim z$ mean that $x^2 - y^2$ and $y^2 - z^2$ are both integers, thus their sum $x^2 - z^2$ is also an integer, which means that $x \sim z$.

(ii) The equivalence class of 2 consists of all those real numbers y for which $4 - y^2$ is an integer, i.e., $y^2 - 4 = a$, i.e, $\{\pm\sqrt{a+4} : a \in \mathbb{Z} \text{ and } a \geqslant -4\}$.

For any x, we have $[x] = \{\pm\sqrt{a+x^2} : a \in \mathbb{Z} \text{ and } a \geqslant -x^2\}$.

3. (a) Hint: $\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2 = \left(\frac{x^2 - y^2}{xy}\right)^2 + 2$.

(b) Straightfoward induction proof.

(c) (i) By contrapositive: we show that if n is not divisible by 3, then n^4 is not divisible by 3.

Indeed, if n is not divisible by 3, then it equals $3k+1$ or $3k+2$ for appropriate k. In the first case,

$$n^4 = (3k+1)^4 = 3(27k^4 + 36k^3 + 18k^2 + 4k) + 1,$$

so it is not divisible by 3. In the second case,

$$n^4 = (3k+2)^4 = 3(27k^4 + 72k^3 + 72k^2 + 32k + 5) + 1,$$

so it is also not divisible by 3. $\square$

(ii) By contradiction: suppose that we can write $\sqrt[4]{3} = a/b$ with $a, b \in \mathbb{Z}$ and $\operatorname{hcf}(a,b) = 1$. Then $a^4 = 3b^4$, so a^4 is a multiple of 3, which by (i) implies that a is a multiple of 3, say, $a = 3k$. But then $(3k)^4 = 3b^4$ which implies that $b^4 = 3(9k^4)$, so b^4 is also a multiple of 3, which again by (i) implies that b is a multiple of 3. This contradicts that $\text{hcf}(a,b)=1$. □

(iii) By contradiction: If the given number is rational, say equal to a/b, then we may express

$$\sqrt[4]{3} = \frac{2a+b}{a-b},$$

which contradicts (ii). □

4. (a) (i) Suppose $A = \{a_1, \ldots, a_{|A|}\}$. Then we can encode the function f by writing out the corresponding element of B as a "word", where the ith letter corresponds to $f(a_i)$:

$$\underset{a_1}{\underline{\quad}} \quad \underset{a_2}{\underline{\quad}} \quad \underset{a_3}{\underline{\quad}} \quad \cdots \quad \underset{a_{|A|}}{\underline{\quad}}$$

in each space, we can put any of the $|B|$ possible output, or 'undefined', so we have $1+|B|$ options for each a_i (since the function doesn't need to be total).

Thus the number of functions is $(1+|B|)(1+|B|)\cdots(1+|B|) = (1+|B|)^{|A|}$. □

(ii) Similar to part (i), but the function must be total, so instead of $|B|+1$, we have $|B|$ choices. Moreover, we cannot repeat ourselves, since f is injective. Thus $|B|(|B|-1)\cdots(|B|-|A|+1)$ is the result, assuming $|A| \leqslant |B|$. If $|A| > |B|$, then the answer is zero.

(b) (i) $[0,15]$ (ii) $\{80\}$ (iii) $[0,3]\cup(8,15]$
(iv) $[-1,8]$ (v) $\{-2,2\}$ (vi) $[-2,2]$

(c) Firstly, f clearly assigns each $x \in (-1,1)$ to a unique real number, so f is total and functional.

Now to see that f is injective, suppose $f(x) = f(y)$. Then:

$$\begin{aligned}
& \frac{x}{\sqrt{1-x^2}} = \frac{y}{\sqrt{1-y^2}} \\
\implies \quad & \frac{x^2}{1-x^2} = \frac{y^2}{1-y^2} \\
\implies \quad & x^2(1-y^2) = y^2(1-x^2) \\
\implies \quad & x^2 = y^2 \\
\implies \quad & y = \pm x
\end{aligned}$$

Clearly $f(x) \neq f(-x)$ since they have opposite signs, so we must have $x = y$, thus f is an injection.

Finally, to see that f is a surjection, let $y \in \mathbb{R}$. Then

$$\begin{aligned}
& f(x) = y \\
\implies \quad & \frac{x}{\sqrt{1-x^2}} = y \\
\implies \quad & x^2 = y^2(1-x^2) \\
\implies \quad & x^2 + x^2y^2 = y^2 \\
\implies \quad & x^2(1+y^2) = y^2 \\
\implies \quad & x = \pm \frac{y}{\sqrt{1+y^2}}
\end{aligned}$$

Of the two, it is clear that we should take the + version, since

$$f\left(\frac{y}{\sqrt{1+y^2}}\right) = \frac{\frac{y}{\sqrt{1+y^2}}}{\sqrt{1-\frac{y^2}{1+y^2}}} = \frac{y}{1-y^2},$$

thus for any $y \in \mathbb{R}$, we can take $x = y/\sqrt{1+y^2}$ to get $f(x) = y$, proving that f is surjective.

Consequently, we see that

$$f^{-1}(x) = \frac{x}{\sqrt{1+x^2}}.$$

(d) (i)
$$\begin{aligned} x \in f^{-1}(A \cup B) &\iff f(x) \in A \cup B && \text{(definition of } f^{-1}\text{)} \\ &\iff f(x) \in A \vee f(x) \in B && \text{(definition } \cup\text{)} \\ &\iff x \in f^{-1}(A) \vee x \in f^{-1}(B) && \text{(definition of } f^{-1}\text{)} \\ &\iff x \in f^{-1}(A) \cup f^{-1}(B) && \text{(definition of } \cup\text{),} \end{aligned}$$
which completes the proof. □

(ii) It is true, and the proof is similar:

$$\begin{aligned} & y \in f(A \cup B) \\ \iff & \exists x : (x \in A \cup B \wedge f(x) = y) && \text{(definition of } f(S)\text{)} \\ \iff & \exists x : ((x \in A \vee x \in B) \wedge f(x) = y) && \text{(definition of } \cup\text{)} \\ \iff & \exists x : (x \in A \wedge f(x) = y) \vee (x \in B \wedge f(x) = y) && \text{(distributivity)} \\ \iff & y \in f(A) \vee y \in f(B) && \text{(definition of } f(S)\text{)} \\ \iff & y \in f(A) \cup f(B), && \text{(definition of } \cup\text{)} \end{aligned}$$
as required. □