L-Università ta' Malta

Department of Mathematics
Faculty of Science

B.Sc. (Hons.) Year I

Sample Examination Paper I

MAT1804: Mathematics for Computing — *n*th January 20XX, 13:00–15:05

Instructions

Read the following instructions carefully.

- Attempt only **THREE** questions.
- Each question carries **35** marks.
- Calculators and mathematical formulæ booklet will be provided.

⚠ Attempt only **THREE** questions.

Question 1.

(a) (i) Express the following in idiomatic English:

$$\forall \mathscr{F} \in \wp\wp\mathbb{N}, (\forall X, Y \in \mathscr{F}, X \cup Y \in \mathscr{F}) \to \exists x \in \mathbb{N} : \frac{\#\{X \in \mathscr{F} : x \in X\}}{\#\mathscr{F}} \geqslant \tfrac{1}{2}.$$

(ii) Write out its negation in symbols, and in English.

(b) Prove the following tautology by constructing a truth table:

$$(\varphi \vee \psi) \wedge (\varphi \to \xi) \wedge (\psi \to \xi) \to \xi.$$

(c) Prove the following set facts.

(i) $A \cup B = A \cup (B \smallsetminus (A \cap B))$ (ii) $A \cap (B \smallsetminus (A \cap B)) = \emptyset$

(iii) $B = (A \cap B) \cup (B \smallsetminus (A \cap B))$ (iv) $(A \cap B) \cap (B \smallsetminus (A \cap B)) = \emptyset$

Hence, using the fact that $|X \cup Y| = |X| + |Y|$ for disjoint sets (i.e., when $X \cap Y = \emptyset$), deduce that for any sets A and B, we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

[7, 4, 24 marks]

Question 2.

(a) Show that the sum of a positive rational number and its reciprocal is always greater than or equal to 2.

(b) Define $f : \mathbb{R} \rightharpoonup \mathbb{R}$ by

$$f(x) = \frac{x}{x-1}$$

for all $x \in \mathbb{R}$ for which the expression makes sense.

(i) Prove that f is an injection.

(ii) What are $\operatorname{ddf}(f)$ and $\operatorname{ran}(f)$?

(iii) Define $g : \operatorname{ddf}(f) \to \operatorname{ran}(f)$ by $g(x) = f(x)$ for all $x \in \operatorname{ddf}(f)$. Show that g is a bijection.

(c) Let $f : X \to Y$ be a function, and let $A, B \subseteq Y$.

(i) Show that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(ii) Show that $f(f^{-1}(A)) \subseteq A$.

(d) A cab to take you back home from a night out with friends costs €P. Some of your friends are thinking of sharing the cab with you, but 3 of them are not sure if they want to leave yet. It is €1 more expensive per person if the 3 friends do not join the ride.

(i) Find a formula for the number of people (including yourself) in terms of the price P.

(ii) The next day you can't remember what the price of the cab was, you think it was either €60 or €72. Which is correct?

[7, 10, 10, 8 marks]

Question 3.

(a) Show by induction that $2^n \cdot 3^{2n} - 1$ is divisible by 17.

(b) (i) Using induction, show that if an integer n is not divisible by 3 then it must equal $3k+1$ or $3k+2$ for some integer k.

(ii) Show that if x^5 is divisible by 3, then x is divisible by 3.

(iii) Hence, show that $\sqrt[5]{3}$ is irrational.

(iv) Deduce that $\dfrac{2+\sqrt[5]{3}}{1-\sqrt[5]{3}}$ is irrational.

(c) Let T be a fixed non-empty subset of a set S. Define the relation $\sim$ on $\wp S$ by $X \sim Y \iff X \cap T = Y \cap T$.

(i) Show that $\sim$ defines an equivalence relation.

(ii) Let $S = \{1,2,3,4,5\}$ and $T = \{1,3\}$.

(A) What is the equivalence class $[\{1,5\}]$?

(B) Describe what $[X]$ is for any $X \in \wp S$.

(d) Prove that the area of an equilateral triangle with base b is $\frac{\sqrt{3}}{4}b^2$.

[8, 12, 8, 5 marks]

Question 4.

(a) (i) For a graph G, show that

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

(ii) Recall that $\Delta(G)$ and $\delta(G)$ are the maximum and minimum degrees of G respectively. Show that

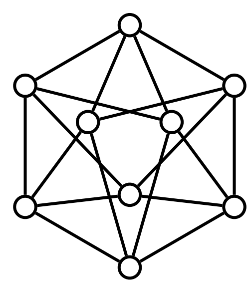
$$\delta(G) \leqslant \frac{2|E(G)|}{|V(G)|} \leqslant \Delta(G).$$

(b) Using induction, show that for any graph G,

$$\chi(G) \leqslant \Delta(G) + 1,$$

where $\chi(G)$ is the chromatic number of G.

(c) Consider the graph G below.



(i) Determine the adjacency matrix of G,

(ii) Find the density of G, (iii) Draw $\overline{G}$.

(d) (i) Let G be a graph on n vertices with no triangles (i.e., K_3 is not a subgraph). Show that for any two adjacent vertices $u, v \in V(G)$, we have $\deg(u) + \deg(v) \leqslant n$, and hence,

$$|E(G)| \leqslant |E(G - u - v)| + |V(G)| - 1.$$

(ii) Hence, use strong induction to deduce Mantel's theorem, which states that if G has no triangles, then

$$|E(G)| \leqslant \frac{|V(G)|^2}{4}.$$

[8, 10, 5, 12 marks]

Answers and Hints

1. (a) (i) For any collection of subsets of the natural numbers, if the union of any two of the sets in the collection is also contained in the collection, then there exists a natural number which is in at least half of the sets of the collection.*

   (ii) $\exists \mathscr{F} \in \wp\wp\mathbb{N}, (\forall X, Y \in \mathscr{F}, X \cup Y \in F) \land \forall x \in \mathbb{N}, \frac{\#\{X \in \mathscr{F} : x \in X\}}{\#\mathscr{F}} < \frac{1}{2}$.

   In words, there is a family $\mathscr{F}$ of subsets of $\mathbb{N}$ such that the union of any two sets in $\mathscr{F}$ remains in $\mathscr{F}$, and every natural number is contained in less than half of the sets of $\mathscr{F}$.

   (b) As usual, you can use `https://lc.mt/tt` to check this.

   (c) (i) Hint: when showing $\subseteq$, add $\land(x \in A \lor x \notin A)$ and expand using distributivity. The $\supseteq$ proof is straightforward.

   (ii) $\supseteq$ is trivial. For $\subseteq$, show that $x \in A \cap (B \smallsetminus (A \cap B)) \to$ false, which shouldn't be hard.

   (iii) A similar approach to the one taken in (i) should work here.

   (iv) Similar to (ii).

   For the final part, we have $|A \cup B| = |A \cup (B \smallsetminus (A \cap B))|$ by (i), which equals $|A| + |B \smallsetminus (A \cap B)|$ by (ii). Then by (iii), we have $|B| = |(A \cap B) \cup (B \smallsetminus (A \cap B))|$, which is $|A \cap B| + |B \smallsetminus (A \cap B)|$ by (iv). Rearranging this, we get that $|B \smallsetminus (A \cap B)| = |B| - |A \cap B|$, which we can plug into the first thing we got to complete the proof.

2. (a) We want to show that $\frac{a}{b} + \frac{b}{a} \geqslant 2$ for all rational $\frac{a}{b} > 0$. This is equivalent to $\frac{a^2+b^2}{ab} \geqslant 2$, which rearranges to $a^2 - 2ab + b^2 \geqslant 0$, i.e., $(a-b)^2 \geqslant 0$, which is clearly true.

   Thus the proof is:

$$(a-b)^2 \geqslant 0 \implies a^2 - 2ab + b^2 \geqslant 0 \implies a^2 + b^2 \geqslant 2ab$$
$$\implies \frac{a^2 + b^2}{ab} \geqslant 2$$
$$\implies \frac{a}{b} + \frac{b}{a} \geqslant 2 \qquad \square$$

*This is the famous union-closed conjecture.

(b) (i) Show that $f(x) = f(y) \Rightarrow x = y$. Indeed, $f(x) = f(y) \Rightarrow \frac{x}{x-1} = \frac{y}{y-1} \Rightarrow x(y-1) = y(x-1) \Rightarrow xy - x = xy - y \Rightarrow x = y$.

(ii) The expression defining f makes sense for all $x \in \mathbb{R}$ except for $x = 1$, thus $\mathrm{ddf}(f) = \mathbb{R} \smallsetminus \{1\}$. Now for the possible values f can take on, let's ask: can $f(x) = r$ for any real number r? Well, $\frac{x}{x-1} = r \Leftrightarrow x = r(x-1) \Leftrightarrow (r-1)x = r \Leftrightarrow x = \frac{r}{r-1}$, which is a meaningful equation so long as $r \neq 1$. Thus $\mathrm{ran}(f) = \mathbb{R} \smallsetminus \{1\}$ also.

(iii) By (i), g is injective, by (ii), g is total & surjective. Thus, g is a bijection.

(c) (i)
$$\begin{aligned} x \in f^{-1}(A \cap B) &\iff f(x) \in A \cap B && \text{(definition of } f^{-1}\text{)} \\ &\iff f(x) \in A \wedge f(x) \in B && \text{(definition } \cap\text{)} \\ &\iff x \in f^{-1}(A) \wedge x \in f^{-1}(B) && \text{(definition of } f^{-1}\text{)} \\ &\iff x \in f^{-1}(A) \cap f^{-1}(B) && \text{(definition of } \cap\text{),} \end{aligned}$$
which completes the proof. □

(ii)
$$\begin{aligned} y \in f(f^{-1}(A)) &\iff \exists x : x \in f^{-1}(A) \wedge f(x) = y && \text{(definition of } f(S)\text{)} \\ &\iff \exists x : f(x) \in A \wedge f(x) = y && \text{(definition of } f^{-1}\text{)} \\ &\implies y \in A && \text{(since } y = f(x)\text{)} \end{aligned}$$
so $f(f^{-1}(A)) \subseteq A$. □

(d) If we let n denote the number of people, then the information given is $\frac{P}{n-3} - \frac{P}{n} = 1$, which we can solve for n to get that

$$n = \tfrac{1}{2}(3 + \sqrt{9 + 12P}).$$

The price was €60 since this gives $n = 15$, whereas €72 gives a non-integer number of people.

3. (a) When $n = 0$, we get 0, which is divisible by 17. For the inductive step, notice that $2^{n+1} \cdot 3^{2(n+1)} - 1 = 2 \cdot 2^n \cdot 3^{2n} \cdot 3^2 - 1 = 18(2^n \cdot 3^{2n}) - 1 = 18(2^n \cdot 3^{2n} - 1 + 1) - 1 = 18(17a + 1) - 1 = 17(18a + 1)$, as required.

(b) (i) Straightforward: in the inductive step, if $n-1$ is divisible by 3, then it equals $3k$ for some k, but then $n = 3k+1$ as required. On the other hand, if $n-1$ is not divisible by 3, then it equals $3k+1$ or $3k+2$ by the IH. In the first case, $n = 3k+2$, as required.

In the second case, $n = 3(k+1)$, so it is divisible by 3 and there is nothing to prove.

(ii) By contrapositive, show that if x is not divisible by 3, then neither is x^5. Use (i) and consider the two cases separately.

(iii) Identical to the proof for $\sqrt{2}$ in the notes, using (iii) instead of the appropriate lemma.

(iv) By contradiction: if this were rational, say equal to a/b, then we could solve for $\sqrt[5]{3}$ and write $\sqrt[5]{3} = \frac{a-2b}{a+b}$, which would be rational, contradicting (iii).

(c) (i) Check reflexivity, symmetry and transitivity, straightforward.

(ii) (A) $[\{1,5\}] = \{\{1\},\{1,2\},\{1,4\},\{1,2,4\},\{1,2,5\},\{1,4,5\},\{1,2,4,5\}\}$

(B) To find $[X]$, first work out $X \cap T$, and then add to it any elements which are not common to T, i.e.,

$$[X] = \{(X \cap T) \cup R : R \subseteq X \text{ and } R \cap T = \emptyset\}.$$

(d) Proof outline: Construct a diagram of an equilateral triangle, and split it into two right-angled triangles with hypotenuse b and base $b/2$. Consequently, we can use Pythagoras' theorem to find the height: $b^2 = (b/2)^2 + h^2$, and then using the usual formula $A = bh/2$, we get the result.

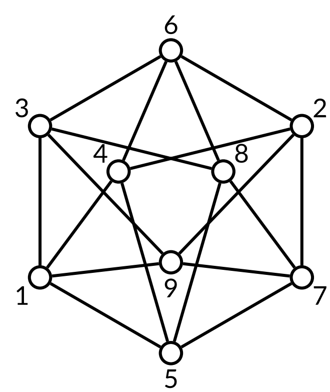
4. (a) Standard from notes.

(b) By induction on the number of vertices of G. Clearly when $|V(G)| = 1$, this is true, since $\chi(G) = 1 \leqslant \Delta(G)+1$. Now fix $v \in V(G)$, and set $G' = G - v$. Clearly $\Delta(G') \leqslant \Delta(G)$. Now by IH, $\chi(G') \leqslant \Delta(G')+1 \leqslant \Delta(G)+1$. Thus we can colour the subgraph G' of G using $\Delta(G)+1$ colours, leaving just v to colour. Since v cannot have more than $\Delta(G)$ neighbours, not all of the $\Delta(G)+1$ of available colours are used by its neighbours, so we can use one of them to colour v. Thus G can be coloured with $\Delta(G)+1$ colours, so $\chi(G) \leqslant \Delta(G)+1$, as required.

(c) (i) This depends on the labelling, so more than one correct answer is possible.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

This is with respect to the following labels:



(ii) $\rho(G) = |E(G)|/\binom{|V(G)|}{2} = 18/\binom{9}{2} = \frac{1}{2}$.

(iii) $\overline{G}$ is the same as G.

(d) (i) If G is triangle free, then any two adjacent vertices cannot have a common neighbour, so $N(u) \cap N(v) = \emptyset$. Thus

$$\begin{aligned} \deg(u) + \deg(v) &= |N(u)| + |N(v)| \\ &= |N(u) \cup N(v)| + |N(u) \cap N(v)| \\ &\leqslant |V(G)| + |\emptyset| = n. \end{aligned}$$

This proves the first part. For the second part, simply observe that when removing u and v, then $\deg(u) + \deg(v) - 1$ edges are deleted (-1 since the edge uv would otherwise be counted twice), thus we get

$$|E(G)| = |E(G-u-v)| + \underbrace{\deg(u)+\deg(v)}_{\leqslant |V(G)|} - 1.$$

(ii) By (strong) induction on the number of vertices. We need two base cases,† when $n=1$, $|E(G)| = 0 \leqslant |V(G)|^2/4 = \frac{1}{4}$, and when $n=2$, we can have at most one edge, so

$$|E(G)| \leqslant 1 = \frac{2^2}{4} = \frac{|V(G)|^2}{4}.$$

Now suppose G has $n>2$ vertices, and remove any two of them obtaining G'. By part (i), $|E(G)| \leqslant |E(G')| + n - 1$, and by IH, $|E(G')| \leqslant (n-2)^2/4$. Thus,

$$E(G) \leqslant \frac{(n-2)^2}{4} + n - 1 = \frac{n^2}{4},$$

as required.

†Because we will be going back two steps in the inductive step, so in the specific case that $n=2$, it's important that *the previous two* are already established.