L-Università ta' Malta

Department of Mathematics
Faculty of Science

B.Sc. (Hons.) Year I

Sample Examination Paper II

MAT1804: Mathematics for Computing

*n*th January 20XX
13:00–15:05

Instructions

Read the following instructions carefully.

- Attempt only **THREE** questions.
- Each question carries **35** marks.
- Calculators and mathematical formulæ booklet will be provided.

⚠ Attempt only **THREE** questions.

Question 1.

(a) (i) State and prove the handshaking lemma.

(ii) Show that in any graph G, $\rho(G) \leqslant \dfrac{\Delta(G)}{|V(G)|-1}$.

(b) Let T be a tree on n vertices.

(i) Show that $|E(T)| = n-1$.

(ii) Suppose that all the vertices are either leaves or have degree d. Show that the number of leaves is

$$\frac{2+(d-2)n}{d-1}.$$

(c) Show that any two longest paths in a connected graph must have a vertex in common.

(d) Show that if G is bipartite, then $|E(G)| \leqslant \dfrac{|V(G)|^2}{4}$.

[*Hint:* $k(n-k)$ *is maximised when* $k = \frac{n}{2}$.]

[9, 12, 7, 7 marks]

Question 2.

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x,y) = (3x+y, 5x+2y).$$

Show that f is bijective, and determine a formula for f^{-1}.

(b) Show by induction that $\displaystyle\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{3n^2-n-2}{4n(n+1)}$.

(c) Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (2x, \frac{1}{x})$. Sketch on an xy-plane:

(i) $f(\{1,2,3\})$ (ii) $f([1,3])$

(iii) $f([-1,1])$ (iv) $f^{-1}([1,3]^2)$

(d) Let $f : X \to Y$ be a function, and let $A, B \subseteq X$.

(i) Show that $f(A \cup B) = f(A) \cup f(B)$.

(ii) Give a counterexample to $f(A \cap B) = f(A) \cap f(B)$.

(iii) Show that if f is injective, then $f^{-1}(f(A)) \subseteq A$.

[10, 8, 8, 9 marks]

Question 3.

(a) Consider the predicates

$$\begin{aligned} o(x) &= \text{“}x \text{ is an odd number”} \\ s(x) &= \text{“}x \text{ is a square number”} \\ p(x,n) &= \text{“}x \text{ can be written as a sum of } n \text{ primes”}. \end{aligned}$$

Translate the following into idiomatic English.

(i) $\forall n \in \mathbb{N}, o(n) \vee s(n)$

(ii) $\forall n \in \mathbb{N}, o(n) \to p(n,3)$

(iii) $\exists m \in \mathbb{N} : \forall n \in \mathbb{N}, p(n,m)$

(iv) $\forall n \in \mathbb{N}, \neg o(n) \wedge s(n) \to p(n,2)$

(b) Give the negations of all the statements in part (a), in symbols and in idiomatic English.

(c) Express the following theorem using symbols only.

> **Theorem (Erdős–Ko–Rado).** *If a collection of subsets of the natural numbers smaller than n is such that the intersection of any two members of the collection is non-empty, and moreover, each set in the collection has cardinality k, where $2k \leqslant n$, then the number of sets in the collection is at most $\binom{n-1}{k-1}$.*

(d) Prove that for any two sets A and B, $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

(e) Construct a truth table for the proposition

$$\varphi \wedge (\psi \vee \xi) \to (\psi \leftrightarrow \neg \xi) \vee \varphi.$$

(f) Write out the elements of the following sets.

(i) $(1,8] \cap 2\mathbb{Z}$

(ii) $\{x \in \mathbb{R} : x^2 < 6x + 7\} \cap \mathbb{Z}$

[4, 4, 4, 10, 8, 5 marks]

Question 4.

(a) (i) Using induction, show that every $n \in \mathbb{N}$ is even or odd, i.e., $n = 2k$ or $n = 2k+1$ for some $k \in \mathbb{N}$.

(ii) Hence, prove that every integer fourth power (n^4) is of the form $8k$ or $8k+1$ for some $k \in \mathbb{N}$.

(b) Show that $n^2 - 9$ is divisible by 8 if and only if n is odd.

(c) Show that if $x^2 + y^2$ is even, then $x + y$ is even.

(d) A real number α is said to be *transcendental* if there does not exist a polynomial f with integer coefficients such that $f(\alpha) = 0$. We know that π and e are both transcendental.

(i) Prove that $\sqrt{2}$ and $\sqrt{3+\sqrt{5}}$ are not transcendental.

(ii) Prove that a transcendental number is irrational.

(iii) It is not known whether the numbers $\pi + e$ and $\pi \times e$ are rational or irrational. Prove that at least one of them is irrational.

[10, 8, 8, 9 marks]

Answers and Hints

1. (a) (i) Standard from notes.

(ii) Recall that $\rho(G)$ is the *density* of the graph, i.e., the number of edges $|E(G)|$ divided by the number of all possible edges, $\binom{|V(G)|}{2} = n(n-1)/2$ where $n = |V(G)|$. Thus

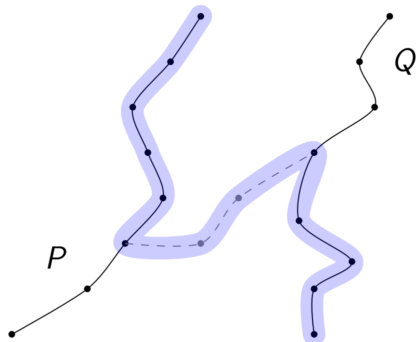
$$\begin{aligned}\rho(G) = \frac{|E(G)|}{|V(G)|(|V(G)|-1)/2} &= \frac{1}{|V(G)|-1} \cdot \frac{2|E(G)|}{|V(G)|} \\ &= \frac{1}{|V(G)|-1} \frac{\sum_{v \in V(G)} \deg(v)}{|V(G)|} \\ &\leqslant \frac{1}{|V(G)|-1} \frac{\sum_{v \in V(G)} \Delta(G)}{|V(G)|} \\ &= \frac{\Delta(G)}{|V(G)|-1},\end{aligned}$$

as required. □

(b) (i) By induction on n. Clearly when $n = 1$ we have $0 = n-1$ edges, which establishes the base case. Now given a tree T on n vertices, remove a leaf ℓ (we know there are always at least 2 leaves) to get $T-\ell$, which by the IH has $(n-1)-1 = n-2$ edges. But adding ℓ back increases the number of edges by 1, so we have $n-1$ edges. □

(ii) Suppose there are k vertices of degree 1 (i.e., leaves). Then there are $n-k$ vertices of degree d, and so the sum of degrees is $k + d(n-k)$, which by the handshaking lemma is $2|E(G)| = 2(n-1)$. Solving the equation $k + d(n-k) = 2(n-1)$ for k gives the desired result. □

(c) By contradiction: suppose there are two longest paths, P and Q, which do not share any vertices. Since the graph is connected, then there is a path from some vertex u of P to some vertex v of Q made up of vertices outside both paths, depicted by the dashed line in the following diagram.



Then by taking the two longer halves of P and Q, together with the path joining them, we get a path in G longer than P or Q, which contradicts that they were longest paths to begin with. □

(d) If G is bipartite, then all edges are between its partite sets, call them A and B. If $|A| = k$, then $|B| = n-k$, and so we can have at most $k(n-k)$ edges. This is largest when $k = n/2$ (which can be seen by differentiation).

Thus $|E(G)|$ surely cannot exceed $\frac{n}{2}(n-\frac{n}{2}) = \frac{n^2}{4}$. □

2. (a) We need to show that f is (i) functional, (ii) total, (iii) injective and (iv) surjective.

(i) Clearly f is functional, since it unambiguously assigns a unique pair of coordinates to each input pair $(x,y) \in \mathbb{R}^2$.

(ii) It is also clear that f is total, since it assigns every point in the domain $\mathbb{R}^2$ a corresponding pair of coordinates.

(iii) To see that f is injective, suppose that $f(x,y) = f(a,b)$, i.e.,

$$(3x+y, 5x+2y) = (3a+b, 5a+2)$$

$$\implies \begin{cases} 3x + y = 3a+b & (1) \\ 5x+2y = 5a+2b & (2) \end{cases}$$

$$\implies \begin{cases} 6x+2y = 6a+2b & 2\cdot(1) \\ 5x+2y = 5a+2b & (2) \end{cases}$$

and subtracting the two equations gives us that $x = a$, and then by (1) we clearly get that $y = b$. Thus, if $f(x,y) = f(a,b)$, then

$(x, y) = (a, b)$, so that f is injective.

(iv) Finally, to see that f is surjective, take any point (x, y) in the codomain $\mathbb{R}^2$, and solve

$$\begin{aligned} & f(a,b) = (x,y) \\ \Longrightarrow \quad & (3a+b, 5a+2b) = (x,y) \\ \Longrightarrow \quad & \begin{cases} 3a + b = x & (1) \\ 5a + 2b = y & (2) \end{cases} \\ \Longrightarrow \quad & \begin{cases} 6a + 2b = 2x & 2\cdot(1) \\ 5a + 2b = y & (2) \end{cases} \end{aligned}$$

subtracting gives $a = 2x - y$, and then using (1), we get that $b = x - 3(2x - y) = 3y - 5x$. Thus we see that

$$f(2x - y, 3y - 5x) = (x, y),$$

and since this works for all (x, y) in the codomain, we see that f is surjective.

We also immediately obtain a formula for f^{-1}, namely,

$$f^{-1}(x, y) = (2x - y, 3y - 5x).$$

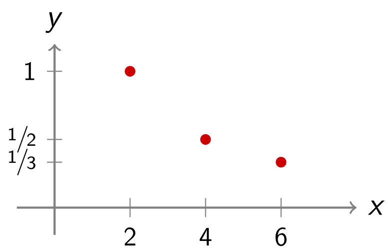
(b) When $n = 2$, we have LHS $= \frac{1}{2^2-1} = \frac{1}{3}$ and RHS $= \frac{3\cdot 2^2-2-2}{4\cdot 2\cdot(2+1)} = \frac{1}{3}$, so the base case is done.

Now for $n > 2$, we have

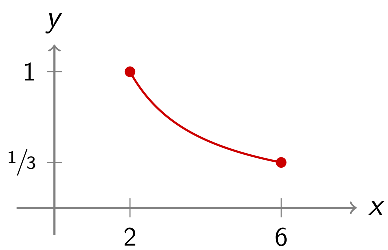
$$\begin{aligned} \sum_{k=2}^{n} \frac{1}{k^2-1} &= \sum_{k=2}^{n-1} \frac{1}{k^2-1} + \frac{1}{n^2-1} \\ &= \frac{3(n-1)^2 - (n-1) - 2}{4(n-1)n} + \frac{1}{n^2-1} \qquad \text{(by IH)} \\ &= \frac{3n^2 - n - 2}{4n(n+1)}, \end{aligned}$$

as required. □

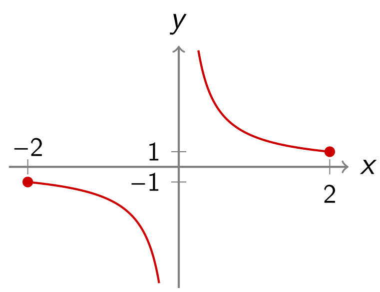
(c) (i)



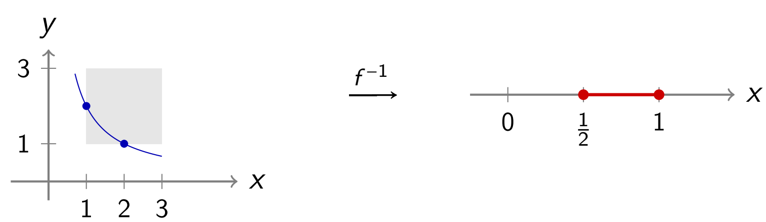
(ii)



(iii)



(iv)



[*Note: the sketch on the left is not necessary, it it there to help understand the sketch on the right*.]

(d) (i) $y \in f(A \cup B)$

$$\begin{aligned}
&\iff \exists x : (x \in A \cup B \land f(x) = y) && \text{(definition of } f(S)) \\
&\iff \exists x : ((x \in A \lor x \in B) \land f(x) = y) && \text{(definition of } \cup) \\
&\iff \exists x : (x \in A \land f(x) = y) \lor (x \in B \land f(x) = y) && \text{(distributivity)} \\
&\iff y \in f(A) \lor y \in f(B) && \text{(definition of } f(S)) \\
&\iff y \in f(A) \cup f(B), && \text{(definition of } \cup)
\end{aligned}$$

as required. □

(ii) Let $X = \{1,2\}$ and $Y = \{1\}$, and define $f : X \to Y$ by $f(x) = 1$ for all $x \in X$. Then if we take $A = \{1\}$ and $B = \{2\}$, we see that $f(A \cap B) = f(\emptyset) = \emptyset$, whereas $f(A) \cap f(B) = \{1\} \cap \{1\} = \{1\}$, so $f(A \cap B) \neq f(A) \cap f(B)$.

(iii) If f is injective, then

$$\begin{aligned}
& x \in f^{-1}(f(A)) \\
\implies & f(x) \in f(A) && \text{(definition of } f^{-1}(S)) \\
\implies & \exists x' : (x' \in A \land f(x') = f(x)) && \text{(definition of } f(S)) \\
\implies & \exists x' : (x' \in A \land x' = x) && \text{(since } f \text{ is injective)} \\
\implies & x \in A && \text{(since } x' = x),
\end{aligned}$$

which shows that $f^{-1}(f(A)) \subseteq A$. □

3. (a) (i) Every natural number is either odd or a square number.

(ii) Every odd natural number can be written as a sum of three primes.

(iii) There is a number m such that every natural number can be expressed as a sum of m primes.

(iv) Every even, square natural number can be expressed as a sum of two primes.

(b) (i) $\exists n \in \mathbb{N} \neg o(n) \land \neg s(n)$
There exists a natural number which is neither odd nor a square.

(ii) $\exists n \in \mathbb{N} : o(n) \land \neg p(n,3)$
There is a natural number which is odd and cannot be written as a sum of three primes.

(iii) $\forall m \in \mathbb{N}, \exists n \in \mathbb{N} : \neg p(n, m)$
For every natural m, there is a natural number which cannot be written as a sum of m primes.

(iv) $\exists n \in \mathbb{N} : \neg o(n) \wedge s(n) \wedge \neg p(n, 2)$
There is an even, square natural number which cannot be written as the sum of two primes.

(c) There is more than one way to do this. Here is a possible way:

$$\forall \mathscr{C} \subseteq \wp(\mathbb{N} \cap [0, n)), \begin{pmatrix} (\forall A, B \in \mathscr{C}, A \cap B \neq \emptyset) \wedge (\forall A \in \mathscr{C}, |A| = k) \\ \wedge (2k \leqslant n) \to |\mathscr{C}| \leqslant \binom{n-1}{k-1} \end{pmatrix}.$$

(d) To show that $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$, we must prove both:

(i) $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$

(ii) $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$

For (i), we have

$$\begin{aligned}
& x \in (A \setminus B) \cup (B \setminus A) & \\
\implies & x \in (A \setminus B) \vee x \in (B \setminus A) & \text{(definition of } \cup\text{)} \\
\implies & (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A), & \text{(definition of } \setminus\text{)} \\
\implies & ((x \in A \wedge x \notin B) \vee x \in B) & \text{(distributivity of} \\
& \quad \wedge ((x \in A \wedge x \notin B) \vee x \notin A), & \vee \text{ over } \wedge\text{)} \\
\implies & ((x \in A \vee x \in B) \wedge (x \notin B \vee x \in B)) & \text{(distributivity of} \\
& \wedge ((x \in A \vee x \notin A) \wedge (x \notin B \vee x \notin A)), & \vee \text{ over } \wedge\text{)} \\
\implies & ((x \in A \vee x \in B) \wedge \text{true}) & \\
& \wedge (\text{true} \wedge (x \notin B \vee x \notin A)), & (\phi \vee \neg\phi \leftrightarrow \text{true}) \\
\implies & (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B), & (\phi \wedge \text{true} \leftrightarrow \phi) \\
\implies & (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B), & \text{(de Morgan's)} \\
\implies & x \in (A \cup B) \wedge x \notin (A \cap B), & \text{(definition of } \cup, \cap\text{)} \\
\implies & x \in (A \cup B) \setminus (A \cap B), & \text{(definition of } \setminus\text{)}
\end{aligned}$$

and for (ii), notice that every step above is reversible (unlike the previous proof), so replacing each $\Rightarrow$ with a $\Leftrightarrow$ gives us a complete proof. □

(e) Use `https://lc.mt/tt` to check this.

(f) (i) $\{2, 4, 6, 8\}$

(ii) $x^2 < 6x + 7 \iff (x+1)(x-7) < 0 \iff -1 < x < 7$ (draw a quick sketch of $y = (x+1)(x-7)$ to understand the last $\Leftrightarrow$).

Thus the answer is $\{0, 1, 2, 3, 4, 5, 6\}$

4. (a) (i) For the base case, $n = 1$ can be expressed as $2 \cdot 0 + 1$.

Now for the inductive step, suppose $n - 1$ can be expressed as $2k$ or $2k + 1$. In the first case, we have $n = 2k + 1$, and in the second case, $n = 2(k+1)$, which completes the proof. $\square$

(ii) By (i), every n can be written as $2k$ or $2k+1$ for some k. Then $(2k)^4 = 16k^4 = 8(2k^4)$, and $(2k+1)^4 = 16k^4 + 32k^3 + 24k^2 + 8k + 1 = 8(2k^4 + 4k^3 + 3k^2 + k) + 1$.

(b) ($\Rightarrow$). If n is odd, then we may express it as $2k+1$ for some k, thus $n^2 - 9 = (2k+1)^2 - 9 = 4k(k+1) - 8$. Now notice that $k(k+1)$ is even since it is the product of two consecutive whole numbers, say it equals 2ℓ. Thus $n^2 - 9 = 8\ell - 8$, which is clearly divisible by 8.

($\Leftarrow$). By contrapositive, if n is not odd, i.e., even, then we may express it as $2k$ for some k, thus $n^2 - 9 = 4k^2 - 10 + 1 = 2(2k^2 - 5) + 1$, which is odd, and therefore cannot be divisible by 8. $\square$

(c) Notice that $x^2 + y^2$ is even if and only if $x^2 + y^2 + 2xy = (x+y)^2$ is even, and $(x+y)^2$ is even if and only if $x + y$ is even.

(d) (i) They are not transcendental since $\sqrt{2}$ is a root of the quadratic $x^2 - 2$ and $\sqrt{3+\sqrt{5}}$ is a root of the quartic $(x^2-3)^2 - 5 = x^4 - 6x^2 - 4$, and both of these have integer coefficients.

(ii) Any rational number a/b is the root of the polynomial $ax - b$ which has integer coefficients, so it cannot be transcendental.

Thus any transcendental number must be irrational.

(iii) e and π are both transcendental, which means that $(x-e)(x-\pi) = x^2 - (e+\pi)x + e\pi$ cannot have coefficients which are all rational. Thus at least one of $e + \pi$ and $e\pi$ is irrational.