L-Università ta' Malta

Department of Mathematics
Faculty of Science

B.Sc. (Hons.) Year I

Sample Examination Paper III

MAT1804: Mathematics for Computing

*n*th January 20XX
13:00–15:05

Instructions

Read the following instructions carefully.

- Attempt only **THREE** questions.
- Each question carries **35** marks.
- Calculators and mathematical formulæ booklet will be provided.

⚠ Attempt only **THREE** questions.

Question 1.

(a) Use strong induction to prove that every integer $n \geqslant 2$ can be written as a product of prime numbers.

(b) Show that if a, b, c are positive real numbers such that $a < b + c$, then

$$\frac{a}{1+a} < \frac{b}{1+b} + \frac{c}{1+c}.$$

(c) (i) The generalised pigeonhole principle states:

If N objects are placed into k boxes, then at least one box contains at least N/k objects.

Prove this by contradiction.

[*Hint: suppose the boxes have size $b_1, \ldots, b_k$, and that each of them is less that N/k, and think about $b_1 + \cdots + b_k$.*]

(ii) Assuming that the Maltese population is 516,000, show that there exist at least two people which have the same birthday and first/last name initials.

(iii) Assuming friendship is mutual (i.e., A is friends with B means that B is friends with A), show that in a group of 20 people there exist two people who have the same number of friends.

[8, 9, 12, 6 marks]

Question 2.

(a) Let S denote the unit-diameter circle with equation $x^2 + y^2 = y$ (tangent at the origin to the x-axis), and let $S' = S \smallsetminus \{(0,1)\}$. Define the function $\pi : S' \to \mathbb{R}$ by

$$\pi(x, y) = \frac{x}{1-y},$$

which maps each point $P \in S'$ to the number $\pi(P)$ on the x-axis, where the ray from $(0,1)$ through the point P intersects the x-axis (see figure 1).

Prove that π is a bijection.

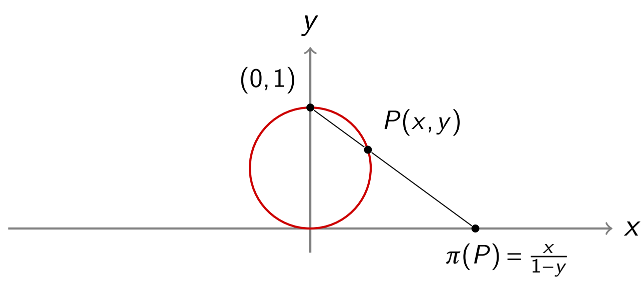

Figure 1: The map $\pi : S' \to \mathbb{R}$

(b) Let A and B be two finite sets.

(i) Show that the number of functions from A to B is $(|B|+1)^{|A|}$.

(ii) How many total injective functions are there from A to B?

(c) Consider the set $X = \{11, 12, \dots, 99\}$, and define the relation $\sim$ on X by $x \sim y \iff x$ and y have the same first digit.

(i) Show that $\sim$ is an equivalence relation.

(ii) What are its distinct equivalence classes?

(d) Let $f : X \rightharpoonup Y$ be a function, and let $A_1, \dots, A_n \subseteq Y$. Show by induction that

$$f^{-1}(A_1 \cup \cdots \cup A_n) = f^{-1}(A_1) \cup \cdots \cup f^{-1}(A_n).$$

[12, 8, 6, 9 marks]

Question 3.

(a) Show that every rational number can be written in the form a/b with $\mathrm{hcf}(a, b) = 1$.

(b) (i) Show using induction that every natural number n is either odd or even, i.e., $n = 2k$ or $n = 2k+1$ for some $k \in \mathbb{N}$.

(ii) Using induction, show that the product of n even numbers is even.

(iii) Deduce that for any $n, m \in \mathbb{N}$, if m^n is odd, then n is odd.

(iv) Show that $\sqrt[m]{2}$ is irrational for every m.

(c) Let $x_1, \dots, x_n$ be positive real numbers satisfying $x_1 + \dots + x_n \leqslant \frac{1}{3}$. Prove by induction that

$$(1-x_1)(1-x_2)\cdots(1-x_n) \geqslant \frac{2}{3}.$$

[*Hint: In the inductive step, consider the* $n-1$ *numbers* $x_1, \dots, x_{n-2}, x_{n-1}+x_n$.]

[10, 16, 9 marks]

Question 4.

(a) Let G be a graph with n vertices and m edges.

(i) Show that

$$\sum_{v \in V(G)} \deg(v) = 2m.$$

(ii) Hence, show that

$$n \cdot \rho(G) \geqslant \delta(G),$$

where $\rho(G)$ and $\delta(G)$ denote the density and the minimum degree of G respectively.

(b) Let G be a connected graph, and let P be a *shortest* path joining the vertices x and y in G. Show that any vertex $v \in V(G)$ cannot have more than 3 neighbours on the path P.

(c) (i) Show that any tree has at least two leaves.

(ii) Prove that the number of edges in a tree on n vertices is $n-1$.

(iii) Show that a tree on n vertices whose degrees are all either 1 or 3 has precisely $\frac{n}{2}+1$ leaves.
[*Hint: use the handshaking lemma*.]

(d) Let G be a k-regular graph on n vertices. Show that

$$\chi(G) \geqslant \frac{n}{n-k}.$$

[*Hint: Consider a* $\chi(G)$*-colouring of* G*, and suppose* n_1 *vertices get colour* 1*,* n_2 *vertices get colour* 2*, ...,* $n_{\chi(G)}$ *get colour* $\chi(G)$*. Then* $n = n_1 + \dots + n_{\chi(G)}$*, and argue that* $n_i \leqslant n-k$ *for all* i.]

[8, 7, 12, 8 marks]