L-Università ta' Malta

Department of Mathematics
Faculty of Science

**B.Sc. IT (Hons.) Year I**

Sample Coursework Assignment

MAT1804: Mathematics for Computing — November 20XX

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# Instructions

This assignment will assess your knowledge of the topics we have covered so far in class, namely propositional logic, set theory and predicate logic.

Read the following instructions carefully.

- This assignment consists of **6 questions** and is out of **100 marks**.
- Submit your solutions in PDF format on VLE by **XXth of November, 20XX**, no later than 12:00.
- Attempt **all** questions.

⚠ This is not a group assignment: plagiarism will not be tolerated.

1. Consider the following propositions.

$$h = \text{"There is a flight from Malta to Heathrow"}$$
$$p = \text{"There is a flight from Malta to Paris"}$$
$$t = \text{"Today is a Tuesday"}$$

   (a) Express the following sentences as propositions in terms of $h$, $p$, $t$.

      (i) There is no flight from Malta to Heathrow.

      (ii) There are flights from Malta to both Heathrow and Paris.

      (iii) If there is a flight from Malta to Paris, there is also one to Heathrow.

      (iv) Unless there is a flight from Malta to Heathrow, then there isn't one to Paris.

      (v) If it's a Tuesday there is either a flight from Malta to Heathrow or to Paris, or both.

      (vi) There are flights to both Heathrow and Paris if it's not a Tuesday.

   (b) Translate the following into idiomatic English.

      (i) $h \wedge \neg p$ (ii) $h \vee p \to h \wedge p$ (iii) $h \wedge t \leftrightarrow \neg p$

      (iv) $h \vee p \wedge t \to h \wedge p \vee t$ (v) $\neg p \wedge \neg t$

**[12, 10 marks]**

2. (a) Construct truth tables for the following. Say which of them (if any) are tautologies or contradictions.

      (i) $\neg\psi \wedge \varphi \vee \psi \to \varphi$ (ii) $\varphi \wedge \psi \vee \varphi \leftrightarrow \varphi$

      (iii) $\neg(\varphi \leftrightarrow \xi) \to \varphi \wedge \neg\xi$ (iv) $\neg(\varphi \wedge \neg(\psi \vee \xi)) \vee \psi$

   (b) You are hungry and arrive late to a party. Luckily your friends offer you some pizza, subject to the condition that once you pick a pizza box, you must stick to it. Unfortunately, since you are late, only one of them has pizza left in it.

   Auspiciously, your friends have written some hints on the pizza boxes with a marker. The only catch is: at least one of the hints


Figure 1: The three pizza boxes

is false, and at least one of them is true. Which box contains the pizza?

**[12, 6 marks]**

3. Consider the following predicates.

$$\begin{aligned} o(x) &= \text{“}x\text{ is an odd number”} \\ p(x) &= \text{“}x\text{ is a prime number”} \\ s(n,m) &= \text{“}n\text{ can be written as a sum of }m\text{ square numbers”} \end{aligned}$$

(a) Translate the following into idiomatic English.

(i) $\forall n \in \mathbb{N}, p(n) \to ((n=2) \vee o(n))$

(ii) $\forall n \in \mathbb{N}, o(n) \wedge p(n) \to s(n,2)$

(iii) $\forall N \in \mathbb{N}, o(N) \to \exists r,s,t \in \mathbb{N} : p(r) \wedge p(s) \wedge p(t) \wedge (N = r+s+t)$

(b) Deduce the negation of each of the statements in part (a), and express it in idiomatic English.

(c) What is the difference between the statements

$$\forall n \in \mathbb{N}, \exists m \in \mathbb{N} : s(n,m) \qquad \text{and} \qquad \exists m \in \mathbb{N}, \forall n \in \mathbb{N} : s(n,m)?$$

Which of them is true (possibly none/both)?
[Hint: You might want to search online for *Bachet's conjecture*].

**[6, 5, 4 marks]**

4. Which of the following are true and which are false?

(a) $3 \in (3, 5]$

(b) $10 \notin (-\infty, \frac{\pi}{2}]$

(c) $7 \in \{2, 3, \ldots, 11\}$

(d) $\pi \in (2, \infty)$

(e) $-1.3 \in \{\ldots, -3, -2, -1\}$

(f) $[1, 2] \subseteq \{0, 1, 2, 3\}$

(g) $\{-1, 0, 1\} \subseteq [-1, 1)$

(h) $[5, 7] \subseteq (4, \infty)$

(i) $\{2, 4, 8, 16, \ldots\} \subseteq [2, \infty)$

(j) $\{\emptyset\} \subseteq X$ for all sets $X$

(k) $\emptyset \subseteq X$ for all sets $X$

(l) $\emptyset \subseteq \wp X$ for all sets $X$

(m) $\{\emptyset\} \subseteq \wp X$ for all sets $X$

(n) $\emptyset \in X$ for all sets $X$

(o) $\emptyset \in \wp X$ for all sets $X$

(p) $\{\{\emptyset\}\} \subseteq \wp\emptyset$

(q) $\{\emptyset\} \subseteq \{\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}\}$

(r) $\wp\{\emptyset\} = \{\emptyset, \{\emptyset\}\}$

**[9 marks]**

5. Consider the set facts

$$A \cap (B \cup A) = A \quad \text{and} \quad (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

(a) Draw a Venn diagram which illustrates the corresponding regions.

(b) Prove the facts from the definitions.

**[4, 20 marks]**

6. Consider the sets

$$A = \{n \in \mathbb{Z} : \exists k \in \mathbb{Z} : n = 2k\} \qquad B = \{n \in \mathbb{Z} : \exists k \in \mathbb{Z} : n = 3k\}$$
$$C = \{n \in \mathbb{Z} : n^2 \text{ is odd}\} \qquad D = \mathbb{Z} \cap [0, 10].$$

Find:

(a) $A \cup C$

(b) $A \cap C$

(c) $A \cap B$

(d) $D \setminus A$

(e) $C \setminus B$

(f) $D \setminus (A \setminus B)$

**[12 marks]**

# Answers

1. (a)
   - (i) $\neg h$
   - (ii) $h \wedge p$
   - (iii) $p \to h$
   - (iv) $h \to p$
   - (v) $t \to h \vee p$
   - (vi) $\neg t \to h \wedge p$

   (b)
   - (i) There is a flight from Malta to Heathrow, but not to Paris.
   - (ii) If there is a flight from Malta to either Heathrow or Paris, then there is a flight to both.
   - (iii) It is Tuesday and there is a flight to Heathrow if and only if there isn't a flight to Paris.
   - (iv) If there's a flight to Heathrow, or it's Tuesday and there's a flight to Paris, then either there's a flight to both Heathrow and Paris, or it's Tuesday.
   - (v) There is neither a flight from Malta to Paris, nor is today Tuesday.

2. (a) You can use `https://lc.mt/tt` to generate truth tables and check your answers. You should find that (ii) is the only tautology.

   (b) The first box contains the pizza.

   *Explanation:* Consider cases. If the pizza is in the first box, then the first two boxes have true statements on them, and the third box has a false one on it. If the pizza is in the second box, all of the statements would be false, whereas if the pizza is in the third box, all the statements would be true. Thus the only possible option is that the pizza is in the first box.

3. (a)
   - (i) Every prime number is either 2, or odd.
   - (ii) Every odd prime can be written as a sum of two squares.
   - (iii) Every odd number can be expressed as the sum of three prime numbers.[*]

   (b)
   - (i) $\exists n \in \mathbb{N} : p(n) \wedge (n \neq 2 \wedge \neg o(n))$ [†]



[*] This is called the Ternary Goldbach conjecture. If you're interested, it was the subject of my master's thesis.

[†] notice I used de Morgan's law $\neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \wedge \neg\psi$ to simplify here.

In English: There is a prime number which is neither 2, nor odd.

(ii) $\exists n \in \mathbb{N} : o(n) \wedge p(n) \wedge \neg s(n,2)$
In English: There is an odd prime which cannot be written as a sum of two squares.

(iii) $\exists N \in \mathbb{N} : o(N) \wedge \forall r,s,t \in \mathbb{N} : \neg p(r) \vee \neg p(s) \vee \neg p(t) \vee N \neq r+s+t$.
In English: There is an odd number which cannot be expressed as the sum of three primes.

(c) The difference is that in the first statement, $m$ can depend on $n$, whereas in the second statement, $m$ is independent of $n$.

The first statement is obviously true, since any natural number can obviously be written as a sum of $n$ squares, namely, as

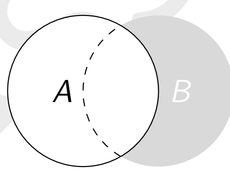
$$n = \underbrace{1^2 + \cdots + 1^2}_{n \text{ times}}.$$

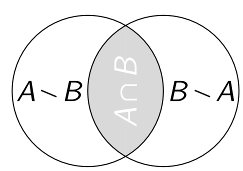
But in the second statement, we claim there exists a value of $m$ which doesn't depend on the value of $n$. With a quick online search, it's easy to come across Lagrange's four-square theorem (less commonly known as Bachet's conjecture), which states that any positive integer can be expressed as the sum of four squares. Thus the second statement is true because $m = 4$ works.

4.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (a) | false | (b) | true | (c) | true | (d) | true | (e) | false | (f) | false |
| (g) | false | (h) | true | (i) | true | (j) | false | (k) | true | (l) | true |
| (m) | true | (n) | false | (o) | true | (p) | false | (q) | false | (r) | true |

5. (a) (i) $A \cap (B \cup A) = A$ (ii) $(A \smallsetminus B) \cup (B \smallsetminus A) = (A \cup B) \smallsetminus (A \cap B)$




☐ in the set ■ not in the set

(b) (i) To show that $A \cap (B \cup A) = A$, we must prove both:

(└) $A \cap (B \cup A) \subseteq A$

(┴) $A \subseteq A \cap (B \cup A)$

For ($\subseteq$), we have

$$\begin{aligned} x \in A \cap (B \cup A) &\implies x \in A \wedge x \in B \cup A && \text{(definition of } \cap\text{)} \\ &\implies x \in A, && (\wedge\text{-elimination}^{\ddagger}) \end{aligned}$$

and for ($\supseteq$), we have

$$\begin{aligned} x \in A &\implies x \in A \wedge (x \in A \vee x \in B) && (\varphi \to \varphi \wedge (\psi \vee \varphi)) \\ &\implies x \in A \wedge x \in A \cup B && \text{(definition of } \cup\text{)} \\ &\implies x \in A \cap (A \cup B), && \text{(definition of } \cap\text{)} \end{aligned}$$

which completes the proof. □

(ii) To show that $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$, we must prove both:

($\subseteq$) $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$

($\supseteq$) $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$

For ($\subseteq$), we have

$$\begin{aligned} & x \in (A \setminus B) \cup (B \setminus A) \\ \implies & x \in (A \setminus B) \vee x \in (B \setminus A) && \text{(definition of } \cup\text{)} \\ \implies & (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A), && \text{(definition of } \setminus\text{)} \\ \implies & ((x \in A \wedge x \notin B) \vee x \in B) && \text{(distributivity of} \\ & \quad \wedge((x \in A \wedge x \notin B) \vee x \notin A), && \quad \vee \text{ over } \wedge) \\ \implies & ((x \in A \vee x \in B) \wedge (x \notin B \vee x \in B)) && \text{(distributivity of} \\ & \wedge((x \in A \vee x \notin A) \wedge (x \notin B \vee x \notin A)), && \quad \vee \text{ over } \wedge) \\ \implies & ((x \in A \vee x \in B) \wedge \text{true}) && \\ & \wedge(\text{true} \wedge (x \notin B \vee x \notin A)), && (\varphi \vee \neg\varphi \leftrightarrow \text{true}) \\ \implies & (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B), && (\varphi \wedge \text{true} \leftrightarrow \varphi) \\ \implies & (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B), && \text{(de Morgan's)} \\ \implies & x \in (A \cup B) \wedge x \notin (A \cap B), && \text{(definition of } \cup, \cap\text{)} \\ \implies & x \in (A \cup B) \setminus (A \cap B), && \text{(definition of } \setminus\text{)} \end{aligned}$$

and for ($\supseteq$), notice that every step above is reversible (unlike the previous proof), so replacing each $\Rightarrow$ with a $\Leftrightarrow$ gives us a complete proof. □

[‡] i.e., $\varphi \wedge \psi \to \varphi$.

6. Notice that $A$ is the set of even numbers, $B$ is the set of multiples of 3, $C$ is the set of odd integers (we saw that $n^2$ is odd if and only if $n$ is odd), and $D$ is just the integers between 0 and 10.

(a) $A \cup C = \mathbb{Z}$

(b) $A \cap C = \emptyset$

(c) $A \cap B = \{n \in \mathbb{Z} : n \text{ is a multiple of } 6\} = \{0, \pm 6, \pm 12, \dots\} = 6\mathbb{Z}$ [§]

(d) $\{1, 3, 5, 7, 9\}$

(e) $\{n \in \mathbb{Z} : n \text{ is odd and not a multiple of } 3\} = \{\pm 1, \pm 5, \pm 7, \pm 11, \dots\}$

(f) $A \smallsetminus B = \{\pm 2, \pm 4, \pm 8, \pm 10, \dots\}$, so $D \smallsetminus (A \smallsetminus B) = \{0, 1, 3, 5, 6, 7, 9\}$.



[§] The notation $n\mathbb{Z}$ means $\{nx : x \in \mathbb{Z}\}$.