Two-Graphs and NSSDs: An Algebraic Approach

Luke Collins Irene Sciriha

DEPARTMENT OF MATHEMATICS
Faculty of Science
University of Malta

Combinatorics and Graph Theory Day
31st January, 2019

Outline

Basic Definitions

Definition (Graph)

A **graph** G is a pair (V, E) where V is a non-empty finite set of **vertices**, and $E \subseteq \binom{V}{2}$ is a set of **edges**, i.e. unordered pairs of the elements of V.

We usually use the letter n for the number of vertices, that is, $n = |V|$.

To encode graphs algebraically, we can use an *adjacency matrix*:

Definition (Adjacency matrix)

The **adjacency matrix** of a graph $G = (V, E)$ is the $n \times n$ matrix (a_{ij}) where

$$a_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j \text{, i.e. } \{v_i, v_j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Seidel Matrix

Another way of encoding graphs is the *Seidel matrix*.

Definition (Seidel matrix)

The **Seidel matrix** of a graph $G = (V, E)$ is the $n \times n$ matrix (s_{ij}) where

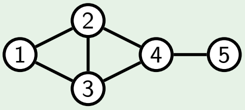
$$s_{ij} = \begin{cases} 0 & \text{if i = j} \\ -1 & \text{if vertex } i \text{ and vertex } j \text{ are adjacent} \\ 1 & \text{otherwise.} \end{cases}$$

Eventually we will work with a variant of the Seidel matrix, which we will introduce later.

Seidel Matrix

Example (A simple Seidel matrix)

Consider the following graph. It has the following Seidel matrix.



$$\Longleftrightarrow \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ -1 & 0 & -1 & -1 & 1 \\ -1 & -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 \end{pmatrix} \end{array}$$

Note that if **A** and **S** are the adjacency and Seidel matrices of G respectively,

$$\mathbf{S} = \mathbf{J} - \mathbf{I} - 2\mathbf{A}$$

where **J** is the matrix consisting entirely of 1's and **I** is the identity matrix.

Spectrum of a Graph

The distinct eigenvalues $\mu_1, \mu_2, \ldots, \mu_s$ of a given matrix $\mathbf{X}$ together with their multiplicities $m_1, m_2, \ldots, m_s$ form the **spectrum** of $\mathbf{X}$, denoted $\mu_1^{(m_1)} \mu_2^{(m_2)} \cdots \mu_s^{(m_s)}$.

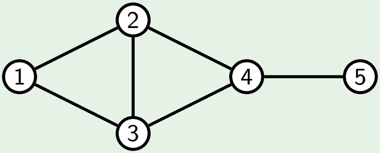
Definition (Spectra)

1. The **spectrum** of a graph G is the spectrum of its adjacency matrix
2. The **Seidel spectrum** of a graph G is the spectrum of its Seidel matrix

Seidel Switching

Given a graph $G = (V, E)$ and a subset of the vertices $U \subseteq V$, the operation of *Seidel switching* with respect to U **exchanges all edges and non-edges** between U and $V \setminus U$ to obtain the graph $\mathrm{SS}(U)$.

Example



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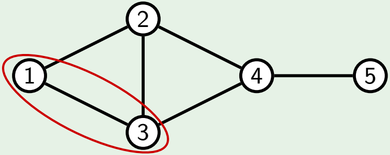

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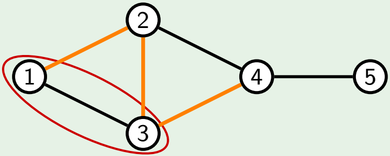

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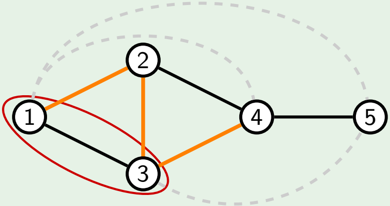

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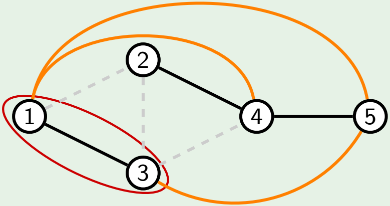

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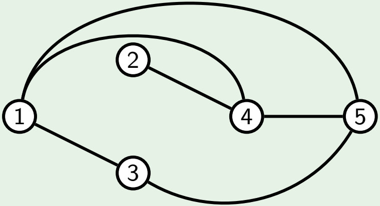
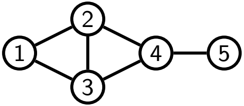


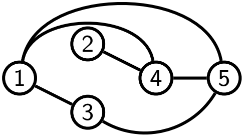
What Seidel Switching does to the Seidel Matrix

We can assume that the vertices of the set $U \subseteq V$ are labelled first (otherwise simply relabel the vertices). In our example, we had the following:

G



$\mathrm{SS}(U)$, where $U = \{1, 3\}$



$$\begin{array}{c} \\ 1 \\ 3 \\ 2 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 3 & 2 & 4 & 5 \end{array} \\ \left(\begin{array}{cc|ccc} 0 & -1 & -1 & 1 & 1 \\ -1 & 0 & -1 & -1 & 1 \\ \hline -1 & -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 \end{array}\right) \end{array} \qquad \begin{array}{c} \\ 1 \\ 3 \\ 2 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 3 & 2 & 4 & 5 \end{array} \\ \left(\begin{array}{cc|ccc} 0 & -1 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 & -1 \\ \hline 1 & 1 & 0 & -1 & 1 \\ -1 & 1 & -1 & 0 & -1 \\ -1 & -1 & 1 & -1 & 0 \end{array}\right) \end{array}$$

What Seidel Switching does to the Seidel Matrix

In general, if $\mathbf{S}$ and $\mathbf{S}_{\mathrm{SS}(U)}$ are the Seidel matrices of G and $\mathrm{SS}(U)$, then

$$\mathbf{S} = \left(\begin{array}{c|c} \mathbf{S}_U & \mathbf{R} \\ \hline \mathbf{R}^\top & \mathbf{S}_{V\setminus U} \end{array}\right) \iff \mathbf{S}_{\mathrm{SS}(U)} = \left(\begin{array}{c|c} \mathbf{S}_U & -\mathbf{R} \\ \hline -\mathbf{R}^\top & \mathbf{S}_{V\setminus U} \end{array}\right).$$

In other words, $\mathbf{S}_{\mathrm{SS}(U)} = \mathbf{D}^{-1}\mathbf{S}\mathbf{D}$, where $\mathbf{D}^{-1} = \mathbf{D}$ is the diagonal matrix with $d_{ii} = +1$ if $i \in U$ and $d_{ii} = -1$ otherwise.

It follows that $\mathbf{S}$ and $\mathbf{S}_{\mathrm{SS}(U)}$ are similar, and therefore G and $\mathrm{SS}(U)$ have the same Seidel spectrum.

Two-Graphs

The operation of Seidel switching defines an equivalence relation on the set of all graphs on n vertices.

Definition (Two-graph)

A **two-graph** or **switching class** is an equivalence class of the Seidel switching equivalence relation.

- A two-graph on n vertices consists of all the n-vertex graphs with the same Seidel spectrum.
- The term 'two-graph' originally arose in a combinatorial context, and actually refers to a couple (V, Δ) where $\Delta \subseteq \binom{V}{3}$ is a collection of triples $\{v_1, v_2, v_3\}$ with the property that any 4-subset of V contains an even number of triples of Δ. This is known to be equivalent to our definition.

Regular Two-Graphs

Definition (Regular two-graph)

A two-graph is said to be **regular** if the Seidel matrix of any representative has precisely two distinct eigenvalues.

- This is a valid definition because the Seidel spectrum of any member of a two-graph is the same.
- Reverting to the combinatorial definition of 'two-graph', (V, Δ) is said to be regular if every pair of vertices lies in the same number of triples of Δ. This is known to be equivalent to our definition.

The Involution $\mathbf{M}$

Suppose $\mathbf{M}$ is a symmetric matrix which is involutionary, that is, $\mathbf{M}^2 = \mathbf{I}$. Then

- By spectral decomposition, $\mathbf{M}$ has eigenvalues 1 and -1.
- If $\mathbf{M}$ is written as

$$\mathbf{M} = \left(\begin{array}{c|c} \mathbf{B} & v \\ \hline v^\top & -\lambda \end{array}\right),$$

then $\mathbf{B}v = \lambda v$ and $|\lambda| < 1$.
Furthermore, if the spectrum of $\mathbf{M}$ is $1^{(n-k)}(-1)^{(k)}$, then it follows by Cauchy's interlacing inequalities that the spectrum of $\mathbf{B}$ is

$$1^{(n-k-1)}(-1)^{(k-1)}\lambda^{(1)}.$$

$\mathbf{M} = \alpha\mathbf{S} - \lambda\mathbf{I}$

If **S** is the Seidel matrix of a regular two-graph on *n* vertices with eigenvalues μ_1, μ_2, then

$$\mathbf{M} = \alpha\mathbf{S} - \lambda\mathbf{I}$$

where

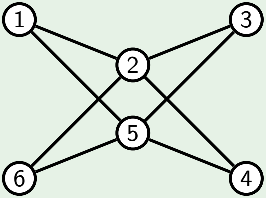
$$\alpha = \frac{2}{\mu_1 - \mu_2} \qquad \text{and} \qquad \lambda = \frac{\mu_1 + \mu_2}{\mu_1 - \mu_2}$$

is an involution.

- This matrix still gives us an encoding of the graph.
- $\mu_1\mu_2 = 1 - n$.

$\mathbf{M} = \alpha\mathbf{S} - \lambda\mathbf{I}$

Example ($K_{2,4}$)



Seidel spectrum: $(-1)^5(5)^1$

$$\alpha = \frac{2}{\mu_1 - \mu_2} = -\frac{1}{3}$$

$$\lambda = \frac{\mu_1 + \mu_2}{\mu_1 - \mu_2} = -\frac{2}{3}$$

$$\mathbf{S} = \begin{pmatrix} 0 & -1 & 1 & 1 & -1 & 1 \\ -1 & 0 & -1 & -1 & 1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 & -1 & 1 \\ -1 & 1 & -1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 1 & -1 & 0 \end{pmatrix}$$

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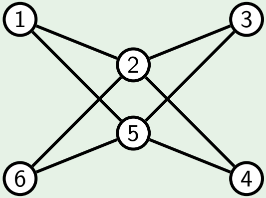

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$$\mathbf{M} = \alpha\mathbf{S} - \lambda\mathbf{I} = \begin{pmatrix} 2/3 & 1/3 & -1/3 & -1/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 & 1/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & 2/3 & -1/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & -1/3 & 2/3 & 1/3 & -1/3 \\ 1/3 & -1/3 & 1/3 & 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & -1/3 & -1/3 & 1/3 & 2/3 \end{pmatrix}$$

Descendant Form

Every two-graph on n vertices has a class representative of the form $D \dot\cup K_1$ where D is a graph on $n-1$ vertices.

Definition (Descendant)

Any two-graph representative of the form $D \dot\cup K_1$ is said to be in *descendant form*, and the component D is said to be a *descendant* of the two-graph.

Obtaining a Descendant Form

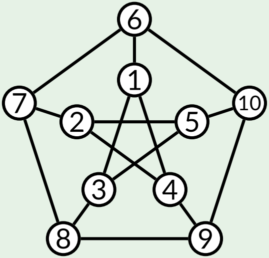
Consider a representative (V, E) which is not in descendant form.

1. Pick any vertex $v \in V$.
2. Let U be the set of all neighbours of v.
3. Then the vertex v is isolated in $\mathrm{SS}(U)$.

Descendant Form

Example (The Petersen Graph)

The famous Petersen graph is contained in a regular two-graph.



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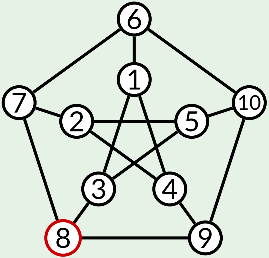

Let us isolate vertex 8.

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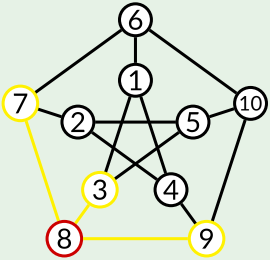

Let us isolate vertex 8. Its set of neighbours is $U = \{7, 3, 9\}$.

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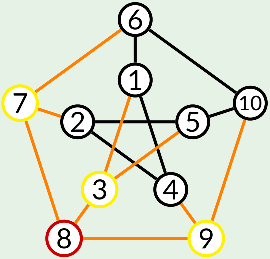

Let us isolate vertex 8. Its set of neighbours is $U = \{7, 3, 9\}$. Now we focus on the edges between U and $V \setminus U$.

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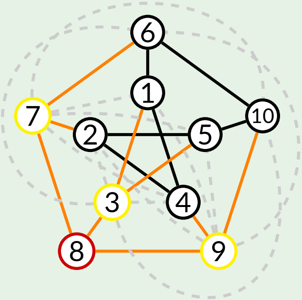

Let us isolate vertex 8. Its set of neighbours is $U = \{7, 3, 9\}$. Now we focus on the edges between U and $V \setminus U$. And the non-edges.

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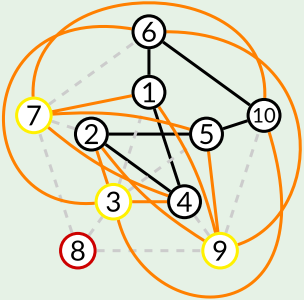

Let us isolate vertex 8. Its set of neighbours is $U = \{7, 3, 9\}$. Now we focus on the edges between U and $V \setminus U$. And the non-edges. Switch edges and non-edges.

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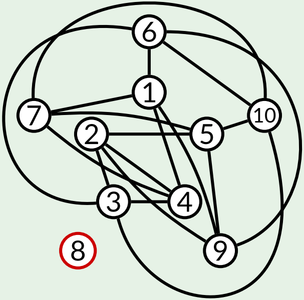

Let us isolate vertex 8. Its set of neighbours is $U = \{7, 3, 9\}$. Now we focus on the edges between U and $V \setminus U$. And the non-edges. Switch edges and non-edges. Obtain $\mathrm{SS}(U)$.

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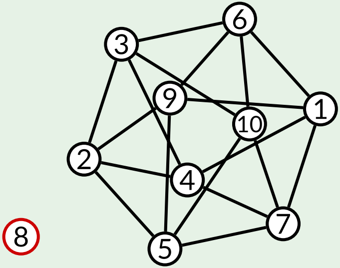


Let us isolate vertex 8. Its set of neighbours is $U = \{7, 3, 9\}$. Now we focus on the edges between U and $V \setminus U$. And the non-edges. Switch edges and non-edges. Obtain $\mathrm{SS}(U)$. Move vertices around to look nicer.

Results about Descendants of Regular Two-Graphs

Using the fact that $\mathbf{M}^2 = \mathbf{I}$, we easily obtain the following known results for descendants of regular two-graphs.

1. D is a ρ-regular subgraph, each vertex having degree

$$\rho = \frac{n}{2} - \frac{\lambda}{\alpha} - 1.$$

2. Substituting for α and λ, we also get that n and $\mu_1 + \mu_2$ have the same parity (even/odd).

Results and Descendants of a Regular Two-Graphs

We prove the first result, that D is ρ-regular with $\rho = \frac{n}{2} - \frac{\lambda}{\alpha} - 1$.

Proof.

Let the Seidel eigenvalues of G be μ_1 and μ_2, where G is in descendant form. Using the values of α and λ, the first and last rows of the involution $\mathbf{M}$ are of the form

$$\begin{array}{r} \text{Row } 1 \\ \\ \text{Row } n \end{array} \begin{pmatrix} -\lambda & \pm\alpha & \pm\alpha & \cdots & \pm\alpha & \alpha \\ & & \vdots & & & \\ \alpha & \alpha & \alpha & \cdots & \alpha & -\lambda \end{pmatrix}$$

where we are assuming that the last row corresponds to the isolated vertex. The number of $-\alpha$'s in row 1 is the degree of vertex 1. Since $\mathbf{M}^2 = \mathbf{I}$, the inner product $\langle \text{Row } 1, \text{Row } n \rangle = 0$.

Results and Descendants of a Regular Two-Graphs

We prove the first result, that D is ρ-regular with $\rho = \frac{n}{2} - \frac{\lambda}{\alpha} - 1$.

Proof.

$$\begin{array}{r} \text{Row 1} \\ \\ \text{Row } n \end{array} \begin{pmatrix} -\lambda & \pm\alpha & \pm\alpha & \cdots & \pm\alpha & \alpha \\ & & \vdots & & & \\ \alpha & \alpha & \alpha & \cdots & \alpha & -\lambda \end{pmatrix}$$

$$\langle \text{Row } 1, \text{Row } n \rangle = 0 \implies -\alpha\lambda - (n-2)\alpha^2 - 2\rho_1\alpha - \alpha\lambda = 0$$

where ρ_1 denotes the degree of vertex 1.
Note that ρ_1 is independent of the vertex label 1, since

$$\langle \text{Row } 1, \text{Row } i \rangle = 0$$

for all $1 \leq i \leq n-1$. Thus D is ρ-regular. □

Strongly Regular Graphs

Recall: A graph is called *regular* if all the vertices are of the same degree.

Definition (Strongly regular graph)

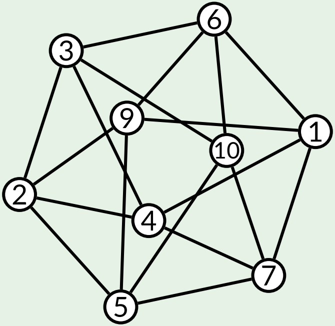
A graph G is said to be a strongly regular graph or an $\operatorname{srg}(n, \rho, e, f)$ if:

1. it has n vertices,
2. each vertex has degree ρ,
3. every two adjacent vertices have e common neighbours, and
4. every two non-adjacent vertices have f common neighbours.

Strongly Regular Graphs

Example (Descendant of Petersen Graph)

The descendant from the last example is an srg$(9, 4, 1, 2)$.



Structure of Descendants of Regular Two-Graphs

Consider a descendant form $D \dot\cup K_1$ of a regular two-graph and the following notations for pairs of vertices.

	# of common neighbours	# of common non-neighbours
Adjacent vertices	$\tilde{e}$	$\tilde{\tilde{e}}$
Non-adjacent vertices	$\tilde{f}$	$\tilde{\tilde{f}}$

By considering the rows of **M** we obtain the following formulæ:

$$\tilde{e} + \tilde{\tilde{e}} = \tfrac{1}{2}(n-2) - \tfrac{\lambda}{\alpha} \qquad \tilde{f} + \tilde{\tilde{f}} = \tfrac{1}{2}(n-2) + \tfrac{\lambda}{\alpha}$$

$$\tilde{e} - \tilde{\tilde{e}} = 2\rho - n \qquad \tilde{f} - \tilde{\tilde{f}} = 2\rho - (n-2)$$

From these it follows that $\tilde{e}$, $\tilde{\tilde{e}}$, $\tilde{f}$ and $\tilde{\tilde{f}}$ are invariant for any pair of adjacent/non-adjacent vertices.

Structure of Descendants of Regular Two-Graphs

From the formulæ obtained previously, we get the following results. Given a descendant form $D \dot{\cup} K_1$ of a regular two-graph on n vertices, then

- D is an $\mathrm{srg}(n-1, \rho, e, f)$ where $e = \tilde{e}$ and $f = \tilde{f} = \frac{\rho}{2}$.
- $\rho = -\frac{1}{2}(1 + \mu_1\mu_2 + (\mu_1 + \mu_2))$ and $e = -\frac{1}{2}(5 + \mu_1\mu_2 + 3(\mu_1 + \mu_2))$.
- n must be even.
- $\frac{\lambda}{\alpha}$ is an integer.

An Application: Conference Graphs

We conclude by mentioning an application of two-graphs. In the paper, we continue to use the theory of NSSD's to study **conference graphs**. These are regular two-graphs which have $\mu_1 = -\mu_2$.

Their Seidel matrices are precisely the so-called **conference matrices**, i.e. $(0, 1, -1)$-matrices with zero on the diagonal and which satisfy $\mathbf{S}\mathbf{S}^\top = k\mathbf{I}$ for some k.

These have applications in telephone networks. A necessary condition for setting up a conference with n telephone ports and ideal signal loss is the existence of an $n \times n$ conference matrix.

An Application: Conference Graphs

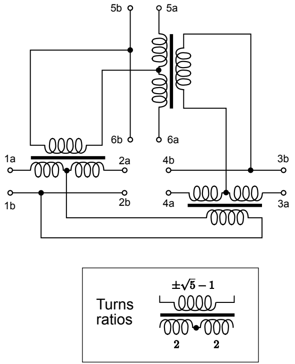


Figure: Implementation of 6-port conference matrix, corresponds to the smallest existing conference graph on $n = 6$ vertices with Seidel eigenvalues $\pm\sqrt{5}$.

Thank you!

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