# The Canonical Double Cover of a Graph

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# Covers

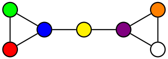
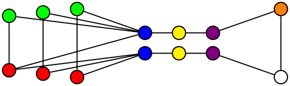
We assume graphs are undirected and have no multiple edges.

## Definition (Cover)

A *cover* of a graph $G$ is another graph $C$ for which there exists a covering map.

A *covering map* is a surjection $f \colon V(C) \to V(G)$ which is a local isomorphism; i.e., $f$ bijectively maps $N(v)$ to $N(f(v))$.

### Example

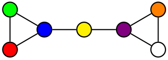
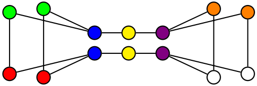
Take $G =$ . Then $C =$ 

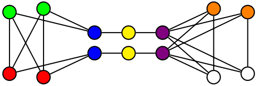
is a cover of $G$, where the corresponding covering map identifies vertices of the same colour.

# Double Covers

A *double cover* is a cover whose corresponding covering map satisfies $|f^{-1}(v)| = 2$ for all $v \in V(G)$.

Example

If we take $G =$ , then $D_1 =$ 

and $D_2 =$  are both double covers of $G$.

In this case, the covering map $f$ is called a *2–1 projection*.

# Canonical Double Cover

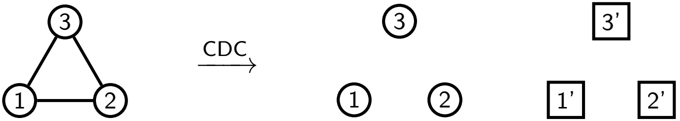
An easy way to get a double cover of a given graph $G$ on the vertices $V(G) = \{1, \dots, n\}$ is to set

$$V(C) = \{1, \dots, n, 1', \dots, n'\}, \quad E(C) = \{uv', u'v : uv \in E(G)\}.$$

Indeed, if we define $f(x) = f(x') = x$ for all $x = 1, \dots, n$, then $f^{-1}(v) = \{v, v'\}$ for all $v \in V(G)$, and by definition, $uv \in E(C)$ if and only if $f(u)f(v) \in E(C)$.

We call this is is called the *canonical double cover* of $G$, $\mathrm{CDC}(G)$. (Equivalently, this is the direct product $G \times K_2$.)

**Example ($\mathrm{CDC}(K_3) = C_6$)**



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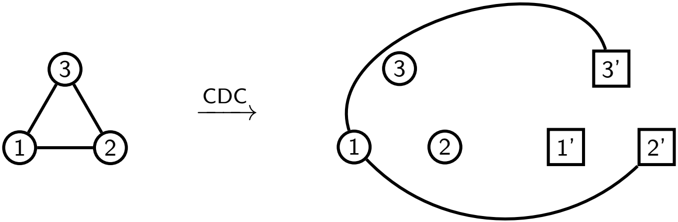

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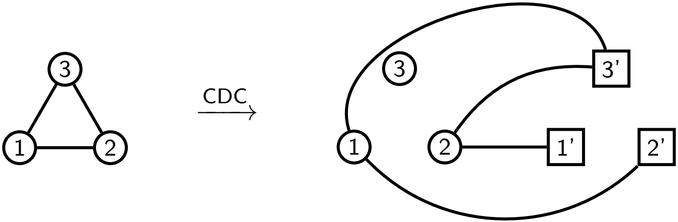

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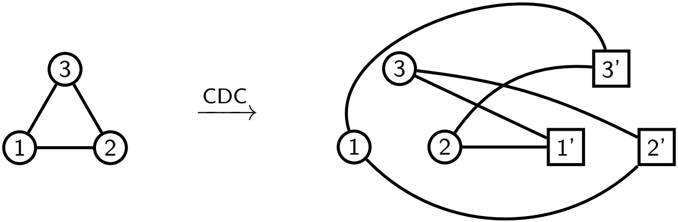

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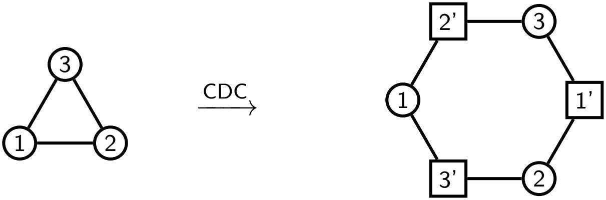
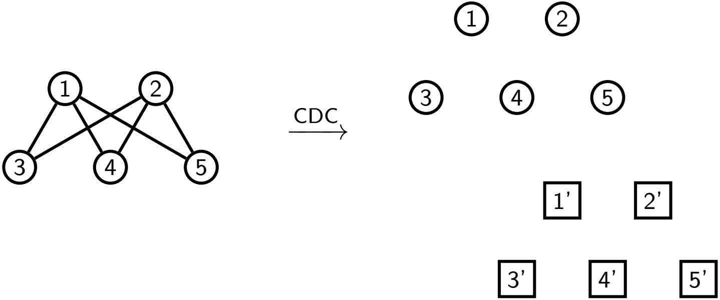


Example ($\mathrm{CDC}(K_{2,3}) = 2K_{2,3}$)



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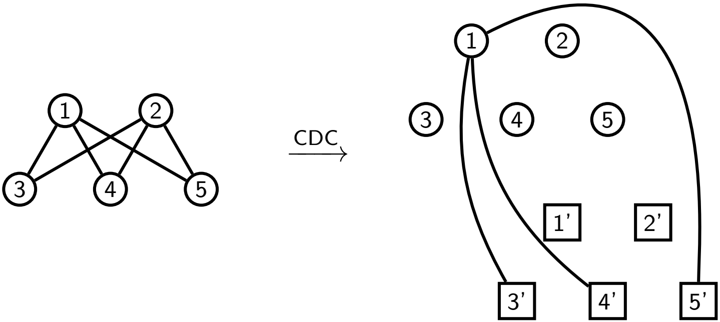

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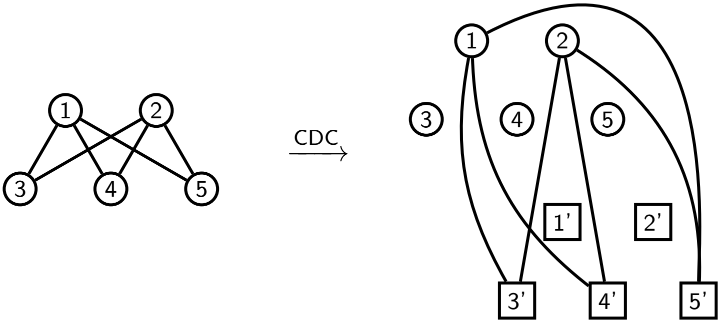

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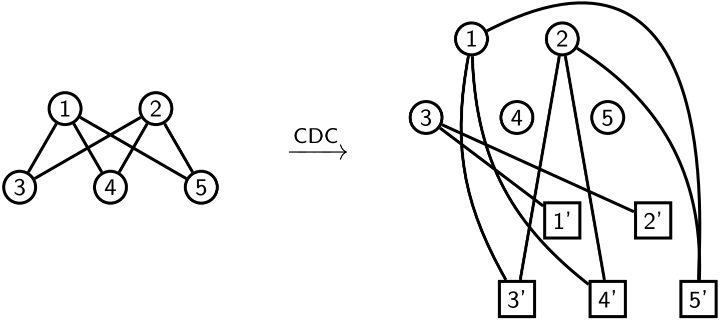

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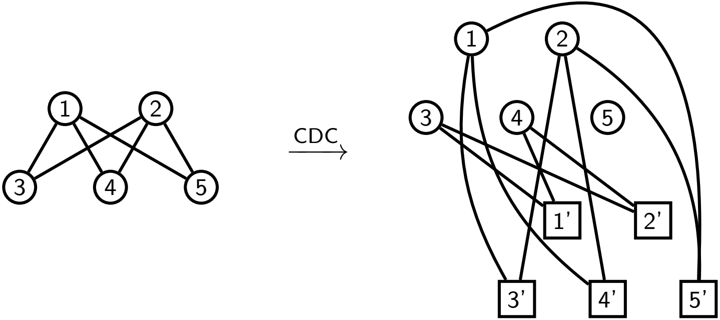

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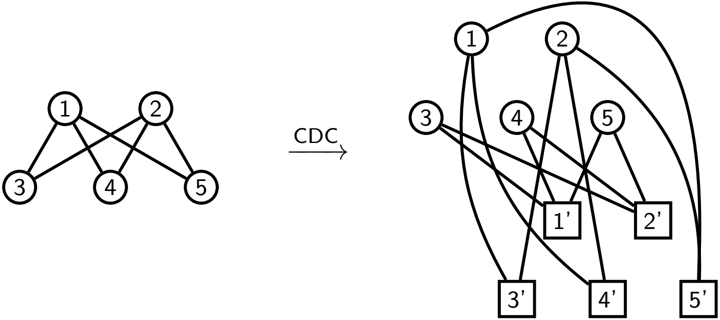

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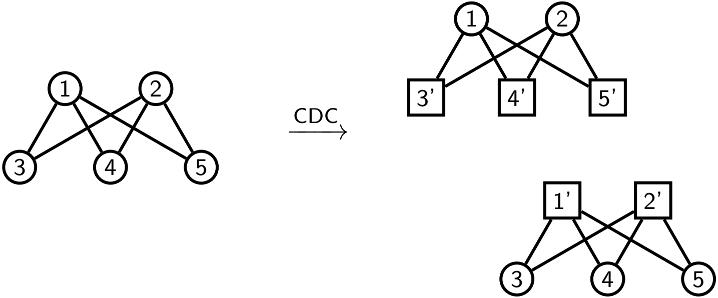


# Some Easy Observations about CDCs

- For any $G$, $\mathrm{CDC}(G)$ is a bipartite graph.
- $\mathrm{CDC}(G+H) = \mathrm{CDC}(G) + \mathrm{CDC}(H)$.
- If $G$ has adjacency matrix $A$, then the adjacency matrix of $\mathrm{CDC}(G)$ is given by

$$\begin{array}{c} \\ 1 \cdots n \\ 1' \cdots n' \end{array} \begin{array}{c} \begin{array}{cc} 1 \cdots n & 1' \cdots n' \end{array} \\ \begin{pmatrix} O & A \\ A & O \end{pmatrix} \end{array}$$

Consequently, the eigenvalues of $C = \mathrm{CDC}(G)$ are $\pm$ those of $G$. Indeed, if $\mathbf{A}_G \boldsymbol{x} = \lambda \boldsymbol{x}$, then

$$\mathbf{A}_C \binom{\boldsymbol{x}}{\boldsymbol{x}} = \lambda \binom{\boldsymbol{x}}{\boldsymbol{x}} \qquad \text{and} \qquad \mathbf{A}_C \binom{-\boldsymbol{x}}{\boldsymbol{x}} = -\lambda \binom{-\boldsymbol{x}}{\boldsymbol{x}}.$$

# Some Easy Observations about CDCs

- Let $G$ be a connected graph. Then

$$\mathrm{CDC}(G) \text{ connected} \iff G \text{ is not bipartite.}$$

**Proof.**

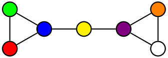
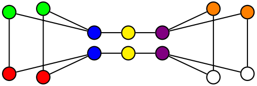
($\Rightarrow$). If $\mathrm{CDC}(G)$ is connected, then there exists a path from 1 to $1'$. But this must alternate between the partite sets: $1 \to v_1' \to v_2 \to \cdots \to v_k \to 1'$ (where $k$ is odd), which corresponds to the odd cycle $1 \to v_2 \to \cdots \to v_k \to 1$ in $G$.

($\Leftarrow$). Suppose $G$ is bipartite with partite sets $U$ and $V$. Let $U, V, U', V'$ denote the corresponding partite sets and their copies in $C = \mathrm{CDC}(G)$. Then $C = C[U \cup V'] + C[U' \cup V] = 2G$. □

Thus for bipartite $G$ we see that $\mathrm{CDC}(G) = 2G$.

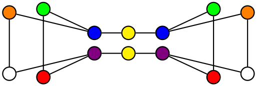
# Our Initial Example

Earlier we looked at the graph

$G =$ , which has $\mathrm{CDC}(G) =$ ,

where the colouring indicates the corresponding 2–1 projection.

But notice that if we recolour as follows:

,

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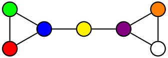
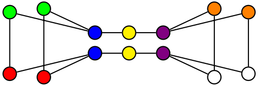
$G =$ , which has $\mathrm{CDC}(G) =$ ,

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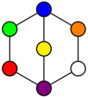
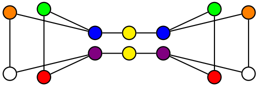
But notice that if we recolour as follows:

$H =$  $\xleftarrow[\text{double cover of}]{}$ ,

It's actually the CDC of $H$! In other words, we found a pair of non-isomorphic graphs $G$ and $H$ such that $\mathrm{CDC}(G) = \mathrm{CDC}(H)$.

## Our Main Question

Suppose $G$ and $H$ are non-isomorphic, and $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$. What can we say about $G$ and $H$?

Before having seen this pair (due to B. Zelinka), it wouldn't have seemed unreasonable to think that $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$ implies $G \simeq H$, since the construction is so simple and preserves so much of the "structure" from the underlying graph.

Indeed, if we have the labelled CDC, all we need to do is identify $x$ and $x'$ with the covering map $f : x, x' \mapsto x$ from earlier.

But an unlabelled CDC might have multiple 2–1 projections to different base graphs.

- The notion of a CDC seems to have been around since 1976 in a paper of Derek Waller (the idea of a "double cover" exists from even before for topological spaces). A popular question previously studied was: given a graph $G$, how many graphs $C$ exist such that $C$ is a double cover of $G$? But this turns out to be quite different from our question.

- Empirically, it is rare for graphs to have the same CDC and not be isomorphic.

| $\lvert V(G)\rvert$ | Non-isomorphic pairs with same CDC |
|---|---|
| 6 | 1 |
| 7 | 4 |
| 8 | 32 |
| 9 | 292 |

(The graph data is available at https://lc.mt/walks.)

- If $\mathsf{CDC}(G) \simeq \mathsf{CDC}(H)$ and $G$ is connected, is $H$ necessarily connected? Answer: **No** since

CDC() =  + ,

and

CDC( + ) = CDC() + CDC() =  + 

- **But** if $\mathsf{CDC}(G) \simeq \mathsf{CDC}(H)$ and $G$ has an isolated vertex, then $H$ must have an isolated vertex as well; i.e.

$$\mathsf{CDC}(G) \simeq \mathsf{CDC}(H) \iff \mathsf{CDC}(G + \circ) \simeq \mathsf{CDC}(H + \circ).$$

Recall that $G$ and $H$ are isomorphic iff $\mathbf{P}\mathbf{A}_G\mathbf{P}^{-1} = \mathbf{A}_H$, where $\mathbf{P}$ is the permutation matrix $(\delta_{i\pi(j)})$ and $\pi\colon V(G) \to V(H)$ is the corresponding isomorphism.

**Definition (Two-fold Isomorphism)**

Let $G$ and $H$ be two graphs. If there exist permutation matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{P}\mathbf{A}_G\mathbf{Q} = \mathbf{A}_H$, then we say that $G$ and $H$ are *two-fold isomorphic*, and write $G \overset{\mathsf{TF}}{\simeq} H$.

In Lauri, Mizzi & Scapellato (2008), the following characterisation is given.

**Theorem**

*For graphs $G$ and $H$, $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \overset{\mathsf{TF}}{\simeq} H$.*

We give a different proof to theirs, using adjacency matrices.

## Theorem

*For graphs $G$ and $H$,* $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \overset{\mathrm{TF}}{\simeq} H$*.*

## Proof.

$(\Longleftarrow)$ If $G$ and $H$ are TF-isomorphic, then by definition there are permutation matrices $R$, $Q$ such that $A_G = RA_HQ$. Then

$$\underbrace{\begin{pmatrix} O & R \\ Q^{\mathsf{T}} & O \end{pmatrix}}_{:=P} \underbrace{\begin{pmatrix} O & A_H \\ A_H & O \end{pmatrix}}_{\mathrm{CDC}(H)} \underbrace{\begin{pmatrix} O & Q \\ R^{\mathsf{T}} & O \end{pmatrix}}_{P^{\mathsf{T}}} = \begin{pmatrix} O & RA_HQ \\ (RA_HQ)^{\mathsf{T}} & O \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} O & A_G \\ A_G & O \end{pmatrix}}_{\mathrm{CDC}(G)}$$

so $\mathrm{CDC}(H) \simeq \mathrm{CDC}(G)$.

## Proof (continued).

( $\Longrightarrow$ ) Suppose $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$. We can assume that both $G$ and $H$ have no isolated vertices, because if they do, we can pair them off. Now since $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$, there exists a permutation matrix $P$ such that

$$P^{\mathsf{T}} \begin{pmatrix} O & A_G \\ A_G & O \end{pmatrix} P = \begin{pmatrix} O & A_H \\ A_H & O \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} P_{11}^{\mathsf{T}} & P_{21}^{\mathsf{T}} \\ P_{12}^{\mathsf{T}} & P_{22}^{\mathsf{T}} \end{pmatrix} \begin{pmatrix} O & A_G \\ A_G & O \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} O & A_H \\ A_H & O \end{pmatrix}$$

Multiplying out and comparing entries, we get that

$$P_{21}^{\mathsf{T}} A_G P_{12} + P_{11}^{\mathsf{T}} A_G P_{22} = A_H \tag{1}$$

$$P_{21}^{\mathsf{T}} A_G P_{11} = P_{12}^{\mathsf{T}} A_G P_{22} = O \tag{2}$$

## Proof (continued).

Now define $Q = (P_{11} + P_{21})^\mathsf{T}$ and $R = P_{22} + P_{12}$. Using the obtained equations (1) and (2), we can expand $QA_G R$ to get

$$QA_G R = A_H. \tag{3}$$

But are $Q$ and $R$ permutation matrices? Suppose not. Being the sum of two submatrices of $P$, this can only happen if a row (and column) are zero, e.g. if

$$P = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

□

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But are $Q$ and $R$ permutation matrices? Suppose not. Being the sum of two submatrices of $P$, this can only happen if a row (and column) are zero.

But by (3) above, $A_H$ will have a row of zeros. This corresponds to an isolated vertex in $H$ — a contradiction. □

Observe that the key to this proof is the contradiction arising from the introduction of an isolated vertex.

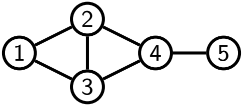
Recall that a **walk** in $G$ is a sequence of vertices

$$v_1, v_2, \ldots, v_k$$

such that $v_i v_{i+1}$ is an edge for $i = 1, \ldots, k-1$. The **length** of a walk is the number $k$ of vertices.

Example

In the graph below, 1234 and 12324 are walks, but 1235 is not.



# Walk Matrix

Let $\mathbf{1} = (1, \dots, 1)$ denote the vector consisting entirely of ones.

**Question:** What is $A\mathbf{1}$ for an adjacency matrix $A$?

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \deg v_1 \\ \deg v_2 \\ \deg v_3 \\ \deg v_4 \\ \deg v_5 \end{pmatrix}$$

What about $A^2\mathbf{1}$?

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \deg v_1 \\ \deg v_2 \\ \deg v_3 \\ \deg v_4 \\ \deg v_5 \end{pmatrix} = \begin{pmatrix} \deg v_2 + \deg v_3 \\ \deg v_1 + \deg v_3 + \deg v_4 \\ \deg v_1 + \deg v_2 + \deg v_4 \\ \deg v_2 + \deg v_3 + \deg v_5 \\ \deg v_4 \end{pmatrix}$$

In general, $A^k\mathbf{1}$ is the vector

$$\begin{pmatrix} \#\text{ of walks of length } k \text{ starting at } v_1 \\ \#\text{ of walks of length } k \text{ starting at } v_2 \\ \vdots \\ \#\text{ of walks of length } k \text{ starting at } v_n \end{pmatrix}.$$

### Definition (Walk Matrix)

The matrix $W_k(G)$ is the $n \times k$ matrix whose columns are the first $k$ such vectors, i.e.

$$W_k(G) = \begin{pmatrix} | & | & | & & | \\ \mathbf{1} & A\mathbf{1} & A^2\mathbf{1} & \cdots & A^{k-1}\mathbf{1} \\ | & | & | & & | \end{pmatrix}.$$

## Theorem

*Let $G$, $H$ be two graphs with $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$, and let $k$ be a natural number. Then*

$$W_G(k) = W_H(k)$$

*for appropriate labelling of the vertices.*

## Proof.

For a graph $\Gamma$, let $A_\Gamma = A(\Gamma)$ and $C_\Gamma = A(\mathrm{CDC}(\Gamma))$. Since $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$, we can relabel the vertices of the graph $H$ to get $H'$, so that $C_G = C_{H'}$. Now for any $0 \leqslant \ell \leqslant k$, we have that

$$C_G{}^\ell \mathbf{1} = \begin{pmatrix} A_G{}^\ell \mathbf{1} \\ A_G{}^\ell \mathbf{1} \end{pmatrix} \qquad \text{and} \qquad C_{H'}{}^\ell \mathbf{1} = \begin{pmatrix} A_{H'}{}^\ell \mathbf{1} \\ A_{H'}{}^\ell \mathbf{1} \end{pmatrix},$$

but since $C_G = C_{H'}$, it follows that $A_G{}^\ell \mathbf{1} = A_{H'}{}^\ell \mathbf{1}$ for all $0 \leqslant \ell \leqslant k$, so the columns of $W_G(k)$ and $W_{H'}(k)$ are equal. $\square$

## Corollary

*If* $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$*, then* $G$ *and* $H$ *have the same degree sequence.*

A nice theorem of Ryser:

## Theorem (Ryser's theorem)

*The graphs* $G$ *and* $H$ *have the same degree sequence iff* $H$ *can be obtained from* $G$ *by a sequence of Ryser switches.*

This means that if $G$ and $H$ have the same CDC, we can obtain $G$ from $H$ by a sequence of such switches. Indeed, on 8 vertices, each of the 32 pairs of graphs require just one Ryser switch, with the exception of $Q_3$ and $2K_4$.

# Conclusion

In the paper we give other results about what CDCs force $G$ and $H$ to have in common, such as their main eigenvalues and eigenspaces. We also introduce the notion of *walk colouring*, this helps to recover a base graph given a CDC.

There are some infinite families of graphs which have the same CDC. Also for certain classes of regular graphs it's best to consider antipodal vertices.

The converse of the walks theorem is false. Indeed, the $k$-walk matrix of a $d$-regular graph is

$$\begin{pmatrix} | & | & | & & | \\ 1 & d & d^2 & \cdots & d^{k-1} \\ | & | & | & & | \end{pmatrix}$$

so regular graphs also have the "same number of walks" property.

- Better understand the nature of these rare pairs having the same CDC.
- Is there is a systematic way to list them exhaustively?

https://lc.mt/walks/same-cdc/

# Thank you!

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