$N!$

Factorials of Large Numbers

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- This talk was going to be about partial summation, a technique which allows us to turn sums into integrals:

$$\sum_{n=1}^{N} f(n) = N\, f(N) - \int_1^N \lfloor t \rfloor\, f'(t)\, dt.$$

- Why? Integrals are easier than sums!

$$\int_1^N \frac{1}{t}\, dt = \log N, \qquad \sum_{n=1}^{N} \frac{1}{n} = \text{¯\\\_(ツ)\_/¯}$$

$$\int_1^N \sqrt{t}\, dt = \frac{2N^{3/2} - 1}{3}, \qquad \sum_{n=1}^{N} \sqrt{n} = \text{¯\\\_(ツ)\_/¯}$$

- The most general form of this idea leads to the famous Euler–Maclaurin summation formula.

- I wrote a blog post on partial summation recently if this idea interests you:

  `https://drmenguin.com/posts/2020/11/partial-summation/`

- I thought it would be more beneficial to illustrate the idea behind this with an example, and a subsequent application:

$$\sum_{n=1}^{N} \log n.$$

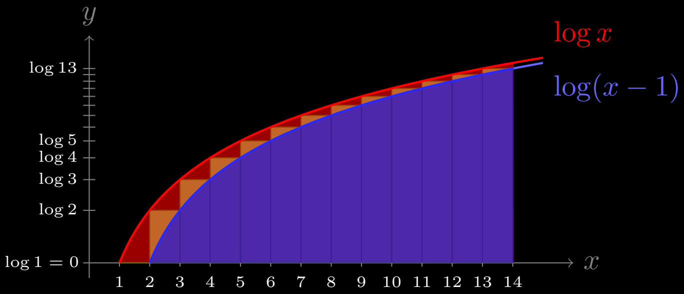
- Observe that

$$\begin{aligned}\sum_{n=1}^{N} \log n &= \log 1 + \log 2 + \cdots + \log N \\ &= \log(1 \cdot 2 \cdots N) \\ &= \log(N!).\end{aligned}$$

  This is progress, but this doesn't really give us much of a feel for the "size" of the sum.

Why should we be able to estimate a sum by an integral?

- The value of our sum is $1 \cdot \log 1 + 1 \cdot \log 2 + \cdots + 1 \cdot \log N$.
- This is the sum of areas of rectangles, each having base 1, and heights $\log 1$, $\log 2$, etc.



- In other words, we have

$$\int_2^N \log(t-1)\,dt \leqslant \sum_{n=1}^{N} \log n \leqslant \int_1^N \log t\,dt$$

Why should we be able to estimate a sum by an integral?

- This allows us to obtain

$$\sum_{n=1}^{N} \log N = N \log N - N + \varepsilon(N),$$

where $\varepsilon(N)$ is an "error" term such that as $N \to \infty$, the relative error $\frac{\varepsilon(N)}{\sum_{n=1}^{N} \log N} \to 0$.

- In other words, as N grows larger, this term becomes less and less significant:

N	$\sum_{n=1}^{N} \log N$	$N \log N - N$	$\frac{\varepsilon(N)}{\sum_{n=1}^{N} \log N}$ (%)
10	15.1044	13.0259	13.76%
1 000	5 912.13	5 907.76	0.07%
1 000 000	12 815 518.38	12 815 510.56	0.00006%

- This is the same result we would obtain if we use partial summation.
- But if we use the Euler–Maclaurin formula instead, we can actually get that

$$\sum_{n=1}^{N} \log n = N \log N - N + \tfrac{1}{2} \log(2\pi N) + \varepsilon(N),$$

but this time, the error $\varepsilon(N) \to 0$ as $N \to \infty$!

- Since we said that our sum is just $\log(N!)$, we therefore have that

$$\begin{aligned} \log(N!) &= N \log N - N + \tfrac{1}{2} \log(2\pi N) + \varepsilon(N) \\ \implies N! &= (e^{\log N})^N \cdot e^{-N} \cdot e^{\log \sqrt{2\pi N}} \cdot e^{\varepsilon(N)} \\ &= \sqrt{2\pi N} \left(\frac{N}{e}\right)^N \cdot e^{\varepsilon(N)} \end{aligned}$$

Stirling's Approximation

$$N! \sim \sqrt{2\pi N}\left(\frac{N}{e}\right)^N.$$

The symbol $\sim$ here does not mean "is approximately equal to" but "is asymptotic to", which means that they are the same as $N \to \infty$.

(Formally, $f \sim g$ means that $\frac{f(N)}{g(N)} \to 1$ as $N \to \infty$.

E.g., $x^2 \sim x^2 + 3x$ but $2x^2 \nsim x^2$.)

We can use Stirling's approximation to get an estimate for, say,

$$10\,000!.$$

This does not simply mean plugging Stirling's approximation with $N = 10\,000$ it into a calculator, it's too big a number!

10 000!

Stirling's formula gives us that

$$10\,000! \approx \sqrt{2\pi\, 10\,000}\left(\frac{10\,000}{e}\right)^{10\,000}.$$

With a bit of massaging, the right-hand side becomes

$$\frac{10^{40\,002}}{e^{10\,000}}\sqrt{2\pi}.$$

Now if we want to write this number in the form $a \times 10^b$ where $a < 10$, notice that $b = \lfloor b + \log_{10} a \rfloor = \lfloor \log_{10}(a \times 10^b) \rfloor$. In other words, taking $\log_{10}$ of the number above will give us the power of 10 we need when we write it in scientific form.

$$\begin{aligned}\log_{10}\left(\frac{10^{40\,002}}{e^{10\,000}}\sqrt{2\pi}\right) &= 40\,002 - 10\,000\log_{10} e + \log_{10}\sqrt{2\pi} \\ &\approx 35\,659.5 \implies b = 35\,659.\end{aligned}$$

10 000!

What about the value of a? This is just $10\,000!/10^{35\,659}$, so by Stirling's approximation again, we get

$$\begin{aligned} a &= 10^{4\,343}\sqrt{2\pi}/e^{10\,000} \\ &= \sqrt{2\pi}\, e^{4\,343 \ln 10 - 10\,000} \\ &\approx \sqrt{2\pi} e^{0.12706} \\ &\approx 2.846239, \end{aligned}$$

Therefore, we get that

$$\boxed{10\,000! \approx 2.846239 \times 10^{35\,659}.}$$

Bonus: de Polignac's formula

Another fun question we can consider.

Clearly 10 000! ends with the digit 0.

How many zeroes does 10 000! end in?

- Notice that each zero corresponds to a factor of 10. Thus we need to find the largest k such that $10^k \mid 10\,000!$.
- A factor of 10 corresponds to the occurrence of a $2 \cdot 5$ in the prime factorisation of 10 000!.
  Since half of the integers between 1 and 10 000 are even, we know the power of 2 in its prime factorisation is at least $10\,000/2 = 5\,000$.

Bonus: de Polignac's formula

Similarly, one fifth of the numbers between 1 and 10 000 are divisible by 5, so the power of 5 is at least $10\,000/5 = 2\,000$.

But some numbers contribute more than one multiple of 5. In fact, one in every 25 numbers contributes two multiples of five, so we need to add an extra $10\,000/25 = 400$.

Similarly, one in every 125 contributes three multiples of 5, so we add an extra $10\,000/125 = 80$.

Continuing this way, we see that, in total, the amount of fives in the prime factorisation is therefore

$$\left\lfloor \frac{10\,000}{5} \right\rfloor + \left\lfloor \frac{10\,000}{25} \right\rfloor + \left\lfloor \frac{10\,000}{125} \right\rfloor + \left\lfloor \frac{10\,000}{625} \right\rfloor + \left\lfloor \frac{10\,000}{3\,125} \right\rfloor = 2\,499.$$

Bonus: de Polignac's formula

For each of these 5's, there is a 2 we can pair them up with (there are many more 2's in fact), but this is the precise number of 5's, so we have that 10 000! ends in $\boxed{2\,499 \text{ zeroes.}}$

de Polignac's formula. The power of the prime number p in the prime factorisation of $N!$ is

$$\sum_{k=1}^{\infty} \left\lfloor \frac{N}{p^k} \right\rfloor = \sum_{k=1}^{\lfloor \log_p N \rfloor} \left\lfloor \frac{N}{p^k} \right\rfloor.$$

E.g. The power of 17 in the prime factorisation of 10 000! is

$$\left\lfloor \frac{10\,000}{17} \right\rfloor + \left\lfloor \frac{10\,000}{17^2} \right\rfloor + \left\lfloor \frac{10\,000}{17^3} \right\rfloor = \boxed{61.}$$

Thank you!

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