eigenvalues in number theory:

Ramanujan Graphs

Luke Collins

Eigenvalue Day
Malta Mathematics Society
University of Malta

29th April, 2021

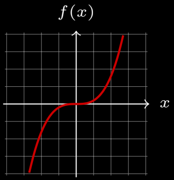
Disclaimer

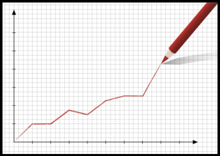
Since I can't get into the technical stuff, this talk is more about graphs than number theory.

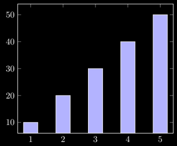
(But the techniques used to study these are number theoretic.)

What is a graph?

In mathematics, a *graph* is not one of these:







What is a graph?

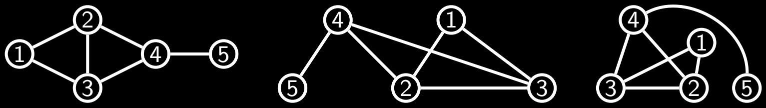
Definition (Graph)

A **graph** G is a pair (V, E) where V is a non-empty finite set, and E is a set of unordered pairs of the elements of V.

The elements of the set V are called *vertices*, and the pairs in E are called *edges*.

Example

$V = \{1, 2, 3, 4, 5\}$ and
$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$ define a graph.



Representing Graphs as Matrices

We usually use the letter n for the number of vertices, that is, $n = |V|$.

To encode graphs algebraically, we can use an *adjacency matrix*:

Definition (Adjacency matrix)

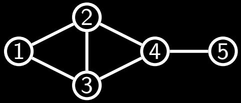
The **adjacency matrix** of a graph $G = (V, E)$ is the $n \times n$ matrix (a_{ij}) where

$$a_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j \text{, i.e. } \{v_i, v_j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Representing Graphs as Matrices

Example (A simple adjacency matrix)

Consider the following graph. It has the following adjacency matrix.



$\Longleftrightarrow$

$$\begin{array}{c@{}c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Note that in general,

- The adjacency matrix is symmetric
- Each 1 represents an edge, and each 0 represents a non-edge
- Each entry on the diagonal is 0, since we consider simple graphs

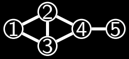
Representing Graphs as Matrices

When we use terminology from linear algebra such as

- *eigenvalues* of a graph,
- *eigenvectors* of a graph,
- *eigenspace* of a graph,
- *column space* of a graph,

and so on, we are actually referring to the **adjacency matrix** of the graph.

Example

The eigenvalues of 

are those of its adjacency matrix $\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, i.e., the roots of $\det(A - \lambda I) = \begin{vmatrix} \lambda & -1 & -1 & 0 & 0 \\ -1 & \lambda & -1 & -1 & 0 \\ -1 & -1 & \lambda & -1 & 0 \\ 0 & 1 & -1 & \lambda & -1 \\ 0 & 0 & 0 & -1 & \lambda \end{vmatrix}$.

Some graph terminology

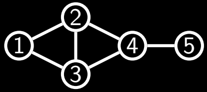
- The *degree* $\deg(v)$ of a vertex v in a graph is how many neighbours it has.
- The *maximum degree* $\Delta(G)$ of a graph is the largest degree in that graph, i.e.,

$$\Delta(G) = \max_{v \in V(G)} \deg(v).$$

- The *distance* $d(u, v)$ between u and v in a graph is the length of a shortest path joining them.
- The *diameter* $\operatorname{diam}(G)$ of a graph G is the longest shortest path, i.e.,

$$\operatorname{diam}(G) = \max_{u,v \in V(G)} d(u, v).$$

Example



Vertex ① has degree 2, vertex ④ has degree 3.
We have $\Delta(G) = 3$.
$d(①, ④) = 2$, $\operatorname{diam}(G) = 3$.

A bound on the eigenvalues of a graph

Theorem

For any eigenvalue λ of G, we have

$$|\lambda| \leqslant \Delta(G).$$

Proof. Let A be the adjacency matrix of the graph G. If λ is an eigenvalue, we have an eigenvector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

If $\boldsymbol{x} = (x_1, \ldots, x_n)$, suppose (wlog) that x_1 is the largest entry. Then

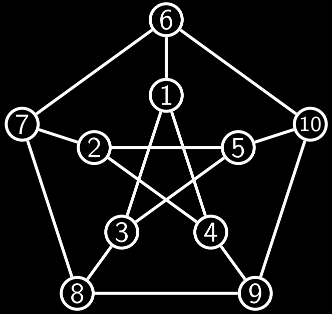
$$\begin{aligned} |\lambda||x_1| &= |\text{the first entry of } A\boldsymbol{x}| \\ &= \Big|\sum_{k=1}^{n} a_{1k}x_k\Big| \leqslant |x_1|\Big|\sum_{k=1}^{n} a_{1k}\Big| = |x_1|\deg(v_1) \\ &\leqslant |x_1|\,\Delta(G), \end{aligned}$$

i.e., $|\lambda| \leqslant \Delta(G)$. □

Regular graphs

A graph is *regular* if all its vertices have the same degree.

Using a similar sort of argument, one can show that k-regular graphs have not only $|\lambda| \leqslant k$ but actually $\lambda = k$ is always an eigenvalue.



For instance, the Petersen graph is 3-regular and has eigenvalues $3, -2, -2, -2, -2, 1, 1, 1, 1, 1$.

$\lambda(G)$

Given a k-regular graph G, we have seen that it definitely has eigenvalue k. (Moreover, if it is bipartite, it has eigenvalue $-k$). Thus eigenvalues $\pm k$ are called *trivial* eigenvalues.

The largest non-trivial eigenvalue of a graph G is denoted by $\lambda(G)$.

For example, the Petersen graph P has $\lambda(P) = 1$.

This eigenvalue tells us quite a bit about the diameter of a k-regular graph. One can prove a number of bounds, such as Chung's 1989 bound

$$\mathrm{diam}(G) \leqslant \frac{\log(n-1)}{\log(k/\lambda(G))} + 1,$$

or the Alon–Boppana bound

$$\lambda(G) \geqslant 2\sqrt{k-1} - o(1) \qquad (\text{as } n \to \infty).$$

Ramanujan Graphs

A *Ramanujan graph* is a k-regular graph G such that

$$\lambda(G) \leqslant 2\sqrt{k-1}.$$

It is difficult to construct graphs which have this property. It is known how to do this for fixed k when $k-1$ is a prime power (when $n \to \infty$), but not in the general case.

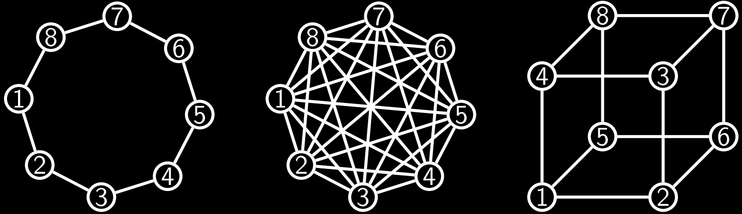
Thus for $k = 7$ we do not know how to construct these, i.e., it is not known how to construct a family of 7-regular graphs whose adjacency matrices have non-trivial eigenvalues $\lambda \leqslant 2\sqrt{6}$ and retains this property as n gets large.

There are very nice constructions of Ramanujan graphs as Cayley graphs of finite groups; the Riemann Hypothesis for curves (proved by Weil) also plays a role in the construction.

Expanders

Ramanujan graphs are of “real world” interest because they make good *expanders*.

Expander graphs are sparse yet have very good connectivity.



It is tempting to quantify connectivity by using the idea of “small diameter”.

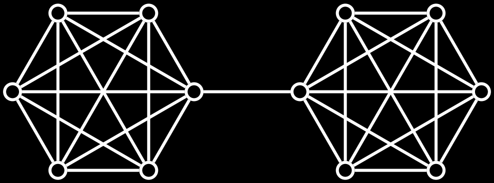
Expanders

Given a subset $U \subseteq V(G)$ of vertices, the boundary ∂U of U is the set of vertices 1 away from the vertices of U, i.e.,

$$\partial U = \{v \in V \setminus U : d(u,v) = 1 \text{ for some } u \in U\}$$

An ε-expander is a graph such that for any subset U of not more than than half the vertices ($|U| \leqslant |V|/2$), we have a decently sized boundary:

$$|\partial U| \geqslant \varepsilon |U|.$$



Thank you!

Luke Collins
luke@collins.mt

Malta Mathematics Society
University of Malta