Walks and Canonical Double Coverings of Comain Graphs

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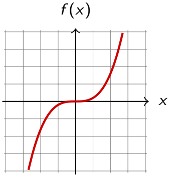
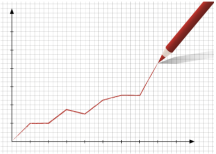
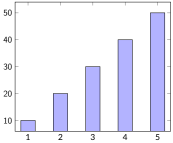
Definition of a Graph

In mathematics, a *graph* is not one of these:







Definition of a Graph

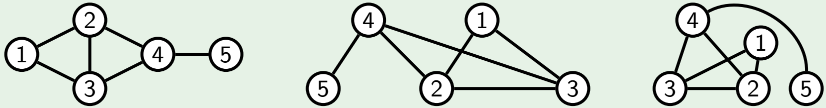
Definition (Graph)

A **graph** G is a pair (V, E) where V is a non-empty finite set, and E is a set of unordered pairs of the elements of V.

The elements of the set V are called *vertices*, and the pairs in E are called *edges*.

Example

$V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$ define a graph.



Representing Graphs as Matrices

We usually use the letter n for the number of vertices, that is, $n = |V|$.

To encode graphs algebraically, we can use an *adjacency matrix*:

Definition (Adjacency matrix)

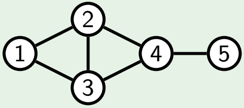
The **adjacency matrix** of a graph $G = (V, E)$ is the $n \times n$ matrix (a_{ij}) where

$$a_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j\text{, i.e. } \{v_i, v_j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Representing Graphs as Matrices

Example (A simple adjacency matrix)

Consider the following graph. It has the following adjacency matrix.



$$\Longleftrightarrow \quad \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Note that in general,

- The adjacency matrix is symmetric
- Each 1 represents an edge, and each 0 represents a non-edge
- Each entry on the diagonal is 0, since we consider simple graphs

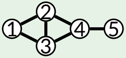
Representing Graphs as Matrices

When we use terminology from linear algebra such as

- *eigenvalues* of a graph,
- *eigenvectors* of a graph,
- *eigenspace* of a graph,
- *column space* of a graph,

and so on, we are actually referring to the **adjacency matrix** of the graph.

Example

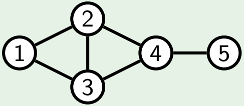
The eigenvalues of 

are those of its adjacency matrix $\begin{pmatrix} 0&1&1&0&0\\ 1&0&1&1&0\\ 1&1&0&1&0\\ 0&1&1&0&1\\ 0&0&0&1&0 \end{pmatrix}$.

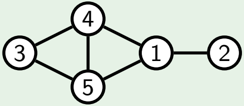
Graph Isomorphisms

Two graphs are *isomorphic* if one can obtain the other by relabelling.

Example



$\simeq$



$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Even though we see that they are the same, their adjacency matrices are completely different!

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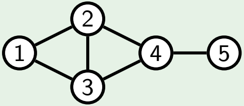

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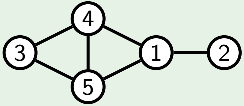


$$\begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4 & 5\end{matrix} \\ \begin{matrix}1\\2\\3\\4\\5\end{matrix} & \begin{pmatrix}0 & 1 & 1 & 0 & 0\\ 1 & 0 & 1 & 1 & 0\\ 1 & 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 0\end{pmatrix}\end{array} = \mathbf{P}^{\mathsf{T}} \begin{array}{cc} & \begin{matrix}1 & 2 & 3 & 4 & 5\end{matrix} \\ \begin{matrix}1\\2\\3\\4\\5\end{matrix} & \begin{pmatrix}0 & 1 & 0 & 1 & 1\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1\\ 1 & 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 1 & 0\end{pmatrix}\end{array} \mathbf{P}$$

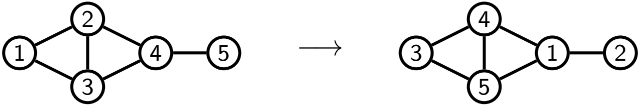
Even though we see that they are the same, their adjacency matrices are completely different! …*or are they?*

Graph Isomorphisms — Permutation Matrices

A *permutation* matrix $\mathbf{P}$ is a matrix obtained from the identity matrix $\mathbf{I}$ by simply rearranging the columns of $\mathbf{I}$. Consequently, they are orthogonal:

$$\mathbf{P}\mathbf{P}^{\mathsf{T}} = \mathbf{I}.$$

In the example the relabelling was: $1 \to 3,\ 2 \to 4,\ 3 \to 5,\ 4 \to 1,\ 5 \to 2$



$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}$$
$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{I}} \longrightarrow \begin{array}{c} 1 \\ \begin{pmatrix} & & 1 & & \\ & & 0 & & \\ & & 0 & & \\ & & 0 & & \\ & & 0 & & \end{pmatrix} \end{array}$$

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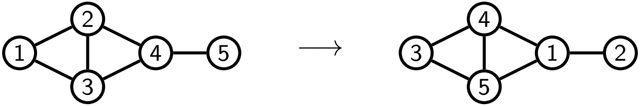

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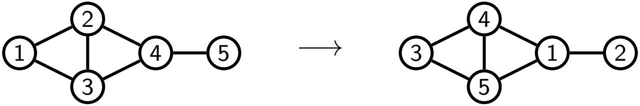

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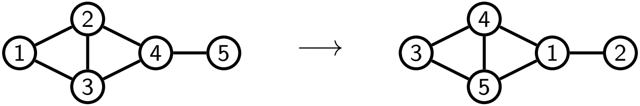

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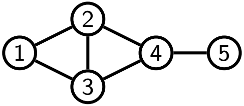
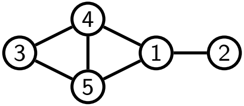


$\longrightarrow$



$$\underset{\mathbf{I}}{\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}} \longrightarrow \underset{\mathbf{P}}{\begin{array}{c} \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}}$$

Graph Isomorphisms — Permutation Matrices

Doing $\mathbf{A} \longrightarrow \mathbf{P}^\top\mathbf{AP}$ will relabel the vertices of the graph of $\mathbf{A}$ according to the corresponding permutation of $\mathbf{P}$.

Moreover, since $\mathbf{P}^\top = \mathbf{P}^{-1}$, we are actually doing $\mathbf{A} \longrightarrow \mathbf{P}^{-1}\mathbf{AP}$. This means that the matrix $\mathbf{A}$ and the resulting new adjacency matrix are *similar*.

Similar matrices have the same:

- eigenvalues
- determinant
- rank
- characteristic polynomial
- minimum polynomial

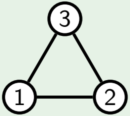
In most cases, the labelling of the vertices in a graph is not important.

Definition of $\mathrm{CDC}(G)$

Definition (Canonical Double Cover)

The **canonical double cover** of a graph $G = (V, E)$ on the vertices $V = \{1, \ldots, n\}$, denoted $\mathrm{CDC}(G)$, is the graph on $2n$ vertices $\{1, \ldots, n, 1', \ldots, n'\}$ whose edges are $\{u, v'\}$ and $\{u', v\}$ for $\{u, v\} \in E$.

Example ($\mathrm{CDC}(K_3) = C_6$)



$\xrightarrow{\mathrm{CDC}}$

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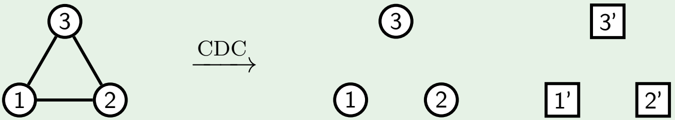

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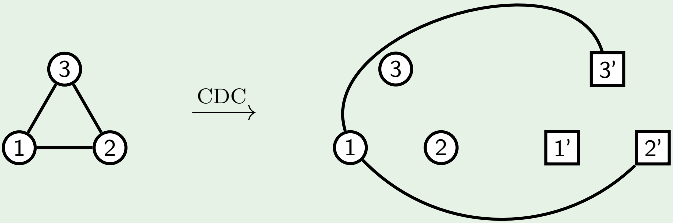

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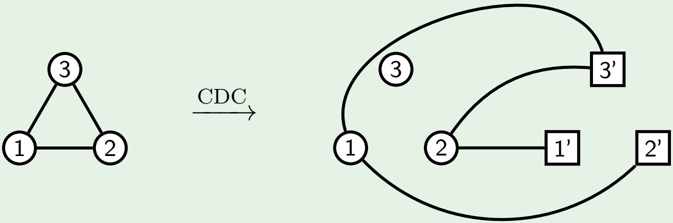

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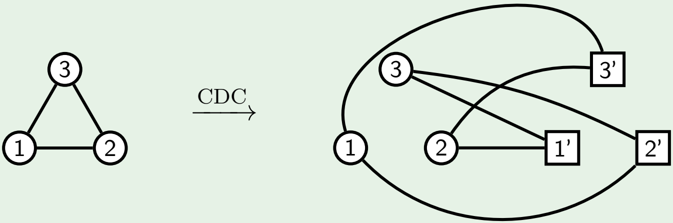

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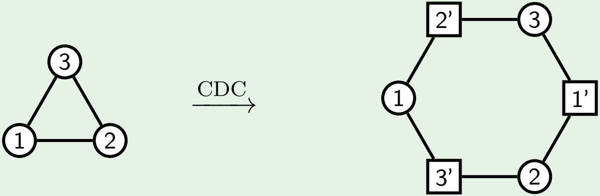
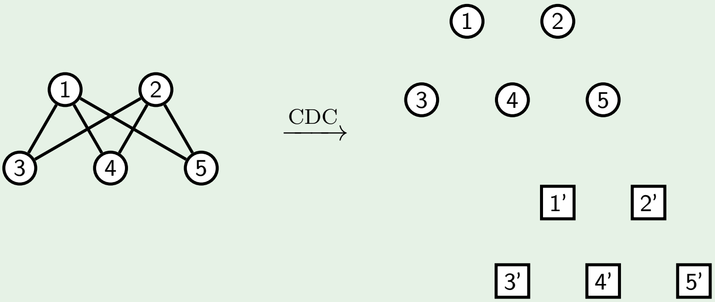


Definition of $\mathrm{CDC}(G)$ — Another example

Example ($\mathrm{CDC}(K_{2,3}) = K_{2,3} \,\dot\cup\, K_{2,3}$)



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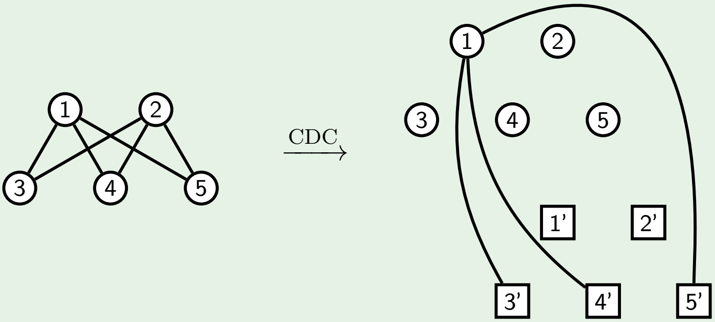

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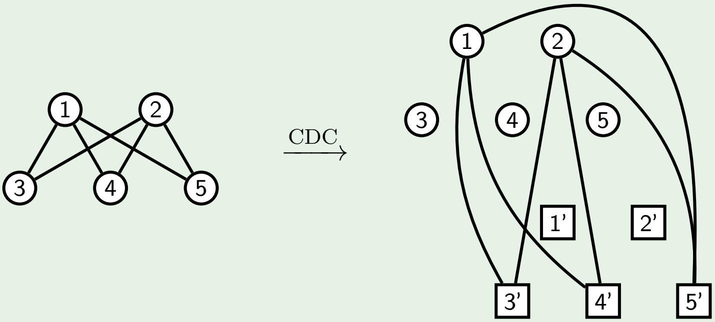

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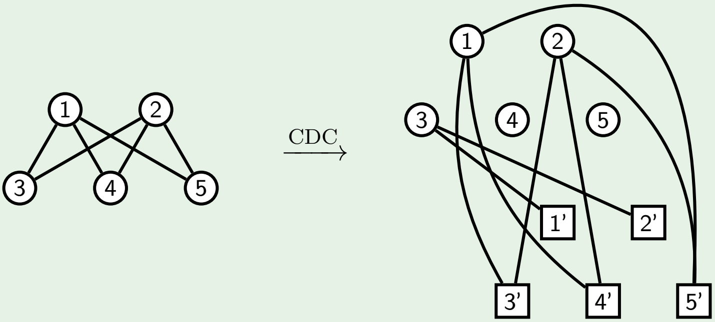

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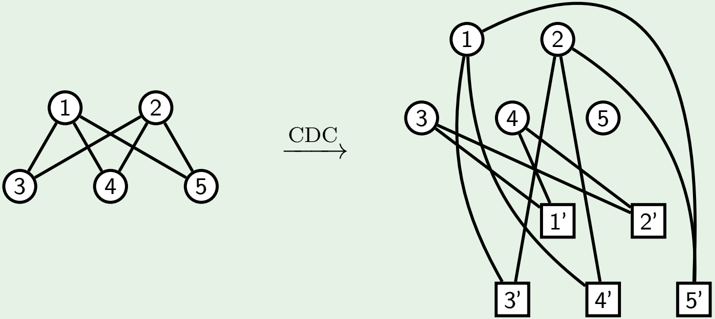

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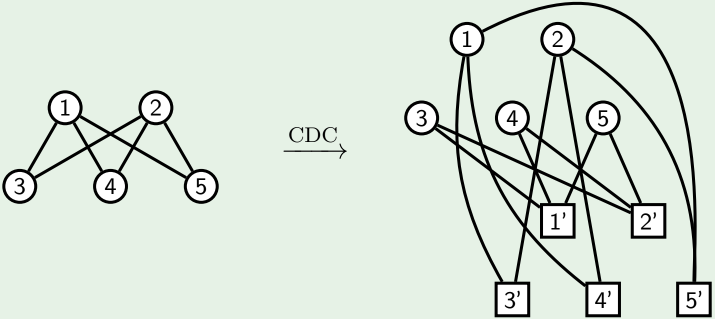

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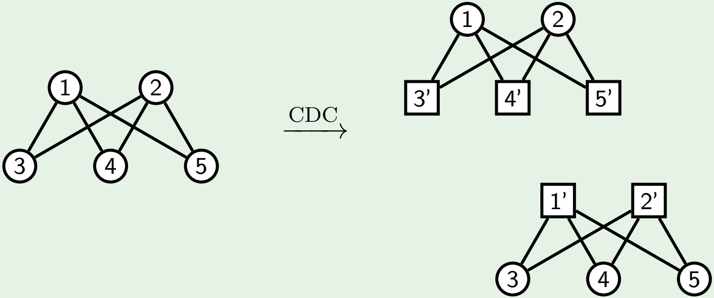


Some Easy Observations about CDCs

- If G has adjacency matrix $\mathbf{A}$, then the adjacency matrix of $\mathrm{CDC}(G)$ is given by

$$\begin{array}{c|cc} & 1\cdots n & 1'\cdots n' \\ \hline 1\cdots n & \mathbf{O} & \mathbf{A} \\ 1'\cdots n' & \mathbf{A} & \mathbf{O} \end{array} \qquad \begin{pmatrix} \mathbf{O} & \mathbf{A} \\ \mathbf{A} & \mathbf{O} \end{pmatrix}$$

- Let G be a connected graph. Then

$$\mathrm{CDC}(G) \text{ connected} \iff G \text{ has an odd cycle},$$

i.e. if G is not bipartite. Moreover, G bipartite $\implies \mathrm{CDC}(G) = G \mathbin{\dot\cup} G$.

- Let $G = G_1 \mathbin{\dot\cup} G_2 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} G_k$. Then

$$\mathrm{CDC}(G) = \mathrm{CDC}(G_1) \mathbin{\dot\cup} \mathrm{CDC}(G_2) \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \mathrm{CDC}(G_k).$$

Some Easy Observations about CDCs

- If $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$ and G is connected, is H necessarily connected? Answer: **No** since

$\mathrm{CDC}($  $) =$  $\dot\cup$ ,

and

$\mathrm{CDC}($  $\dot\cup$  $) = \mathrm{CDC}($  $) \,\dot\cup\, \mathrm{CDC}($  $) =$  $\dot\cup$ 

by the previous result about CDCs of graph unions.

- **But** if $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$ and G has an isolated vertex, then H must have an isolated vertex as well; i.e.

$$\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff \mathrm{CDC}(G \,\dot\cup\, \circ) \simeq \mathrm{CDC}(H \,\dot\cup\, \circ)$$

This yielded a useful *proof technique*: If we have $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$ where G has no isolated vertex, we show that the negation of what we want to prove introduces an isolated vertex in H.

Walks

Definition (Walk)

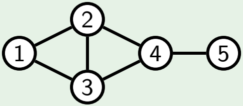
Let G be a graph. A **walk** in G is a sequence of vertices

$$v_1, v_2, \ldots, v_k$$

such that $\{v_i, v_{i+1}\}$ is an edge for $i = 1, \ldots, k-1$. The **length** of a walk is the number k of vertices.

Example

In our usual example graph, 1234 and 12324 are walks, but 1235 is not.



Walk Matrix

Let $\mathbf{j} = (1, 1, \ldots, 1)$ be a vector consisting entirely of ones.

Question: What is $\mathbf{Aj}$ for an adjacency matrix $\mathbf{A}$?

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \deg v_1 \\ \deg v_2 \\ \deg v_3 \\ \deg v_4 \\ \deg v_5 \end{pmatrix}$$

What about $\mathbf{A}^2\mathbf{j}$?

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \deg v_1 \\ \deg v_2 \\ \deg v_3 \\ \deg v_4 \\ \deg v_5 \end{pmatrix} = \begin{pmatrix} \deg v_2 + \deg v_3 \\ \deg v_1 + \deg v_3 + \deg v_4 \\ \deg v_1 + \deg v_2 + \deg v_4 \\ \deg v_2 + \deg v_3 + \deg v_5 \\ \deg v_4 \end{pmatrix}$$

Walk Matrix

In general, $\mathbf{A}^k\mathbf{j}$ is the vector

$$\begin{pmatrix} \#\text{ of walks of length } k \text{ starting at } v_1 \\ \#\text{ of walks of length } k \text{ starting at } v_2 \\ \vdots \\ \#\text{ of walks of length } k \text{ starting at } v_n \end{pmatrix}.$$

Definition (Walk Matrix)

The matrix $\mathbf{W}_G(k)$ is the $n \times k$ matrix whose columns are the first k such vectors, i.e.

$$\mathbf{W}_G(k) = \begin{pmatrix} | & | & | & & | \\ \mathbf{j} & \mathbf{A}\mathbf{j} & \mathbf{A}^2\mathbf{j} & \cdots & \mathbf{A}^{k-1}\mathbf{j} \\ | & | & | & & | \end{pmatrix}.$$

Theorem

Let G, H be two graphs with $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$, and let k be a natural number. Then

$$\mathbf{W}_G(k) = \mathbf{W}_H(k)$$

for appropriate labelling of the vertices.

Proof.

Define $\mathbf{A}_\Gamma = \mathbf{A}(\Gamma)$ and $\mathbf{C}_\Gamma = \mathbf{A}(\mathrm{CDC}(\Gamma))$. Since $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$, we can relabel the vertices of the graph H to get H', so that $\mathbf{C}_G = \mathbf{C}_{H'}$. Now for any $0 \le \ell \le k$, we have that

$$\mathbf{C}_G{}^\ell \mathbf{j} = \begin{pmatrix} \mathbf{A}_G{}^\ell \mathbf{j} \\ \mathbf{A}_G{}^\ell \mathbf{j} \end{pmatrix} \qquad \text{and} \qquad \mathbf{C}_{H'}{}^\ell \mathbf{j} = \begin{pmatrix} \mathbf{A}_{H'}{}^\ell \mathbf{j} \\ \mathbf{A}_{H'}{}^\ell \mathbf{j} \end{pmatrix},$$

but since $\mathbf{C}_G = \mathbf{C}_{H'}$, it follows that $\mathbf{A}_G{}^\ell \mathbf{j} = \mathbf{A}_{H'}{}^\ell \mathbf{j}$ for all $0 \le \ell \le k$, so the columns of $\mathbf{W}_G(k)$ and $\mathbf{W}_H(k)$ are equal. $\square$

Main Eigenspace

An eigenvalue μ of a graph G is said to be *main* if its corresponding eigenspace

$$\mathcal{E}(\mu) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mu\mathbf{x}\}$$

is not entirely orthogonal to **j**, i.e. if $\mathcal{E}(\mu) \not\subseteq \{\mathbf{j}\}^\perp$.

Since **A** is real-symmetric, then $\mathbb{R}^n = \bigoplus_\mu \mathcal{E}(\mu)$.

Consider the eigenspaces $\mathcal{E}(\mu)$ for main eigenvalues μ. Take the projection $\mathbf{x}_\mu := \pi_\mu(\mathbf{j})$ of **j** onto this eigenspace as an initial basis vector, and perform the Gram-Schmidt orthogonalisation process. This yields an orthogonal basis for $\mathcal{E}(\mu)$ with only $\mathbf{x}_\mu$ being not orthogonal to **j**. This is called the *principal main eigenvector* corresponding to μ.

Main Eigenspace

Definition (Main Eigenspace)

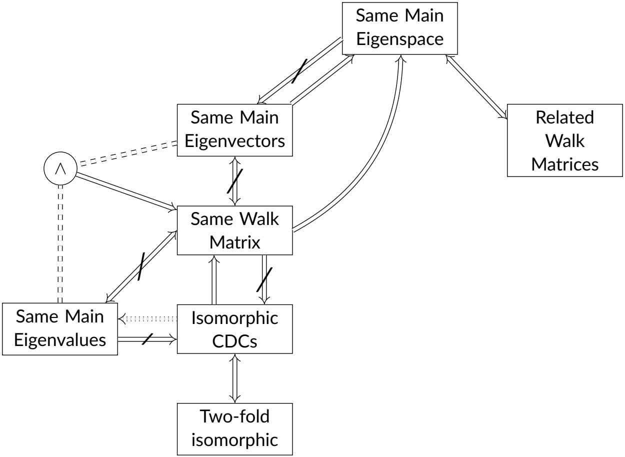
Let G be a graph. The *main eigenspace* of G is the space generated by the principal main eigenvectors of G:

$$\operatorname{Main}(G) = \operatorname{span}\{\pi_{\mu_1}(\mathbf{j}), \dots, \pi_{\mu_p}(\mathbf{j})\},$$

where $\mu_1, \dots, \mu_p$ are the main eigenvalues of G.

Results

In my research, the following hierarchy of graph relations is established.



TF-Isomorphisms (Lauri et al.)

Let us illustrate the CDC proof technique with the following result. Recall that two graphs G and H with adjacency matrices $\mathbf{A}_G$ and $\mathbf{A}_H$ are isomorphic if and only if there exists a permutation matrix such that

$$\mathbf{A}_G = \mathbf{P}^\mathsf{T}\mathbf{A}_H\mathbf{P}.$$

If we weaken this relationship, we get the following:

Definition (Two-Fold Isomorphism)

Let G and H be two graphs with adjacency matrices $\mathbf{A}_G$ and $\mathbf{A}_H$. We say that G is *two-fold isomorphic* or *TF-isomorphic* to H if

$$\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$$

for some permutation matrices $\mathbf{R}$, $\mathbf{Q}$.

Here $\mathbf{R}$ and $\mathbf{Q}$ could be any permutation matrices, they don't have to be the inverse (i.e. transpose) of each other.

Theorem

Let **G** *and* **H** *be two graphs. Then*

$$\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \textit{ and } H \textit{ are TF-isomorphic}.$$

Proof.

($\Longleftarrow$) If G and H are TF-isomorphic, then by definition there are permutation matrices $\mathbf{R}$, $\mathbf{Q}$ such that $\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$. Then

$$\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^{\mathsf{T}} & \mathbf{O} \end{pmatrix}$$

Theorem

Let **G** *and* **H** *be two graphs. Then*

$$\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic.}$$

Proof.

($\Longleftarrow$) If G and H are TF-isomorphic, then by definition there are permutation matrices $\mathbf{R}$, $\mathbf{Q}$ such that $\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$. Then

$$\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^\mathsf{T} & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix}$$

Theorem

Let **G** *and* **H** *be two graphs. Then*

$$\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic.}$$

Proof.

($\Longleftarrow$) If G and H are TF-isomorphic, then by definition there are permutation matrices $\mathbf{R}$, $\mathbf{Q}$ such that $\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$. Then

$$\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^\mathsf{T} & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R}^\mathsf{T} & \mathbf{O} \end{pmatrix}$$

Theorem

*Let **G** and **H** be two graphs. Then*

$$\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic.}$$

Proof.

($\Longleftarrow$) If G and H are TF-isomorphic, then by definition there are permutation matrices $\mathbf{R}$, $\mathbf{Q}$ such that $\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$. Then

$$\underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^\mathsf{T} & \mathbf{O} \end{pmatrix}}_{:=\mathbf{P}} \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R}^\mathsf{T} & \mathbf{O} \end{pmatrix}$$

Theorem

Let $\mathbf{G}$ *and* $\mathbf{H}$ *be two graphs. Then*

$$\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \textit{ and } H \textit{ are TF-isomorphic}.$$

Proof.

($\Longleftarrow$) If G and H are TF-isomorphic, then by definition there are permutation matrices $\mathbf{R}$, $\mathbf{Q}$ such that $\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$. Then

$$\underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^\top & \mathbf{O} \end{pmatrix}}_{:=\mathbf{P}} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix}}_{\mathrm{CDC}(H)} \begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R}^\top & \mathbf{O} \end{pmatrix}$$

Theorem

*Let **G** and **H** be two graphs. Then*

$$\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \textit{ and } H \textit{ are TF-isomorphic.}$$

Proof.

($\Longleftarrow$) If G and H are TF-isomorphic, then by definition there are permutation matrices $\mathbf{R}$, $\mathbf{Q}$ such that $\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$. Then

$$\underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^\mathsf{T} & \mathbf{O} \end{pmatrix}}_{:=\mathbf{P}} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix}}_{\mathrm{CDC}(H)} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R}^\mathsf{T} & \mathbf{O} \end{pmatrix}}_{\mathbf{P}^\mathsf{T}}$$

Theorem

Let **G** *and* **H** *be two graphs. Then*

$$\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic}.$$

Proof.

($\Longleftarrow$) If G and H are TF-isomorphic, then by definition there are permutation matrices $\mathbf{R}$, $\mathbf{Q}$ such that $\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$. Then

$$\underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^\mathsf{T} & \mathbf{O} \end{pmatrix}}_{:=\mathbf{P}} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix}}_{\mathrm{CDC}(H)} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R}^\mathsf{T} & \mathbf{O} \end{pmatrix}}_{\mathbf{P}^\mathsf{T}} = \begin{pmatrix} \mathbf{O} & \mathbf{R}\mathbf{A}_H\mathbf{Q} \\ (\mathbf{R}\mathbf{A}_H\mathbf{Q})^\mathsf{T} & \mathbf{O} \end{pmatrix}$$

Theorem

Let **G** *and* **H** *be two graphs. Then*

$$\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \textit{ and } H \textit{ are TF-isomorphic.}$$

Proof.

($\Longleftarrow$) If G and H are TF-isomorphic, then by definition there are permutation matrices $\mathbf{R}$, $\mathbf{Q}$ such that $\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$. Then

$$\underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^\top & \mathbf{O} \end{pmatrix}}_{:=\mathbf{P}} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix}}_{\mathrm{CDC}(H)} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R}^\top & \mathbf{O} \end{pmatrix}}_{\mathbf{P}^\top} = \begin{pmatrix} \mathbf{O} & \mathbf{R}\mathbf{A}_H\mathbf{Q} \\ (\mathbf{R}\mathbf{A}_H\mathbf{Q})^\top & \mathbf{O} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{O} & \mathbf{A}_G \\ \mathbf{A}_G & \mathbf{O} \end{pmatrix}$$

Theorem

Let **G** *and* **H** *be two graphs. Then*

$$\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \textit{ and } H \textit{ are TF-isomorphic}.$$

Proof.

($\Longleftarrow$) If G and H are TF-isomorphic, then by definition there are permutation matrices $\mathbf{R}$, $\mathbf{Q}$ such that $\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$. Then

$$\underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^\mathsf{T} & \mathbf{O} \end{pmatrix}}_{:=\mathbf{P}} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix}}_{\mathrm{CDC}(H)} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R}^\mathsf{T} & \mathbf{O} \end{pmatrix}}_{\mathbf{P}^\mathsf{T}} = \begin{pmatrix} \mathbf{O} & \mathbf{R}\mathbf{A}_H\mathbf{Q} \\ (\mathbf{R}\mathbf{A}_H\mathbf{Q})^\mathsf{T} & \mathbf{O} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{A}_G \\ \mathbf{A}_G & \mathbf{O} \end{pmatrix}}_{\mathrm{CDC}(G)}$$

Theorem

Let **G** *and* **H** *be two graphs. Then*

$$\mathrm{CDC}(G) \simeq \mathrm{CDC}(H) \iff G \text{ and } H \text{ are TF-isomorphic.}$$

Proof.

($\Longleftarrow$) If G and H are TF-isomorphic, then by definition there are permutation matrices $\mathbf{R}$, $\mathbf{Q}$ such that $\mathbf{A}_G = \mathbf{R}\mathbf{A}_H\mathbf{Q}$. Then

$$\underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q}^\mathsf{T} & \mathbf{O} \end{pmatrix}}_{:=\mathbf{P}} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix}}_{\mathrm{CDC}(H)} \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{Q} \\ \mathbf{R}^\mathsf{T} & \mathbf{O} \end{pmatrix}}_{\mathbf{P}^\mathsf{T}} = \begin{pmatrix} \mathbf{O} & \mathbf{R}\mathbf{A}_H\mathbf{Q} \\ (\mathbf{R}\mathbf{A}_H\mathbf{Q})^\mathsf{T} & \mathbf{O} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \mathbf{O} & \mathbf{A}_G \\ \mathbf{A}_G & \mathbf{O} \end{pmatrix}}_{\mathrm{CDC}(G)}$$

so $\mathrm{CDC}(H) \simeq \mathrm{CDC}(G)$.

Proof (continued).

($\implies$) Suppose $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$. We can assume that both G and H have no isolated vertices, because if they do, we can pair them off. Now since $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$, there exists a permutation matrix $\mathbf{P}$ such that

$$\mathbf{P}^\mathsf{T} \begin{pmatrix} \mathbf{O} & \mathbf{A}_G \\ \mathbf{A}_G & \mathbf{O} \end{pmatrix} \mathbf{P} = \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix}$$

$$\implies \begin{pmatrix} \mathbf{P}_{11}^\mathsf{T} & \mathbf{P}_{21}^\mathsf{T} \\ \mathbf{P}_{12}^\mathsf{T} & \mathbf{P}_{22}^\mathsf{T} \end{pmatrix} \begin{pmatrix} \mathbf{O} & \mathbf{A}_G \\ \mathbf{A}_G & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{O} & \mathbf{A}_H \\ \mathbf{A}_H & \mathbf{O} \end{pmatrix}$$

Multiplying out and comparing entries, we get that

$$\mathbf{P}_{21}^\mathsf{T}\mathbf{A}_G\mathbf{P}_{12} + \mathbf{P}_{11}^\mathsf{T}\mathbf{A}_G\mathbf{P}_{22} = \mathbf{A}_H \tag{1}$$

$$\mathbf{P}_{21}^\mathsf{T}\mathbf{A}_G\mathbf{P}_{11} = \mathbf{P}_{12}^\mathsf{T}\mathbf{A}_G\mathbf{P}_{22} = \mathbf{O} \tag{2}$$

Proof (continued).

Now define $\mathbf{Q} = (\mathbf{P}_{11} + \mathbf{P}_{21})^\mathsf{T}$ and $\mathbf{R} = \mathbf{P}_{22} + \mathbf{P}_{12}$. Using the obtained equations (1) and (2), we can expand $\mathbf{QA}_G\mathbf{R}$ to get

$$\mathbf{QA}_G\mathbf{R} = \mathbf{A}_H. \tag{3}$$

But are $\mathbf{Q}$ and $\mathbf{R}$ permutation matrices? Suppose not. Being the sum of two submatrices of $\mathbf{P}$, this can only happen if a row (and column) are zero, e.g. if

$$\mathbf{P} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

□

Proof (continued).

Now define $\mathbf{Q} = (\mathbf{P}_{11} + \mathbf{P}_{21})^{\mathsf{T}}$ and $\mathbf{R} = \mathbf{P}_{22} + \mathbf{P}_{12}$. Using the obtained equations (1) and (2), we can expand $\mathbf{QA}_G\mathbf{R}$ to get

$$\mathbf{QA}_G\mathbf{R} = \mathbf{A}_H. \tag{3}$$

But are $\mathbf{Q}$ and $\mathbf{R}$ permutation matrices? Suppose not. Being the sum of two submatrices of $\mathbf{P}$, this can only happen if a row (and column) are zero, but by (3) above, $\mathbf{A}_H$ will have a row of zeros. This corresponds to an isolated vertex in H— a contradiction. $\square$

Observe that the key to this proof is the contradiction arising from the introduction of an isolated vertex.

How many different graphs have $\mathrm{CDC}(G) \simeq \mathrm{CDC}(H)$?

We have seen that two graphs being TF-isomorphic and having isomorphic CDCs are equivalent.

Question: How many non-isomorphic graphs have the same CDCs?

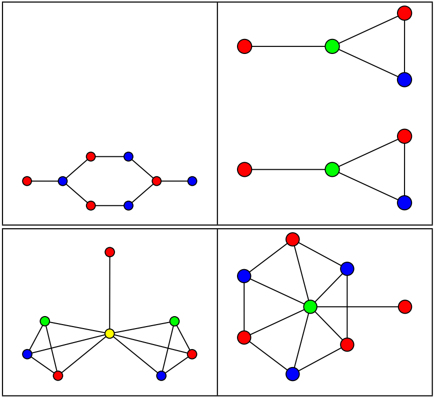
Having the same CDC is very "close" to being isomorphic. On $n \leqslant 8$ vertices, there are 13 597 non-isomorphic graphs. Taking all

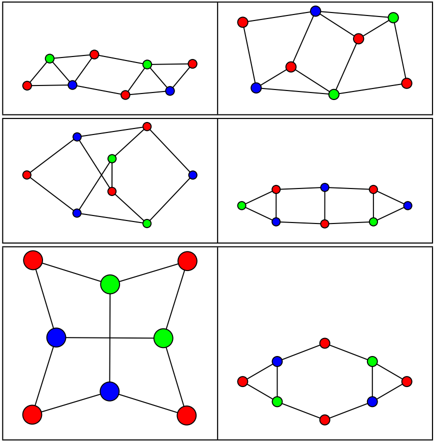
$$\binom{13\,597}{2} = 92\,432\,406$$

possible pairs of graphs on at most 8 vertices, it turns out that only 32 pairs are TF-isomorphic.

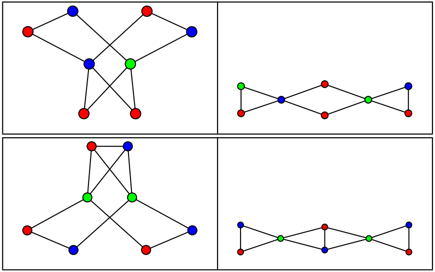
It is rare for a pair of graphs to be so structurally similar yet not isomorphic.

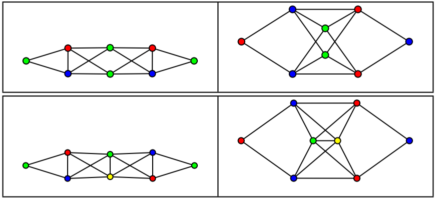
Here are some of them:





and some more:





Thank you!

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