L-Università ta' Malta

Department of Mathematics
Faculty of Science

B.Sc. (Hons.) Year I

Semester I Examination Session 2024/25

MAT1804: Mathematics for Computing — 4th February 2025, 8:30–10:35

Instructions

Read the following instructions carefully.

- Attempt only **THREE** questions.
- Each question carries **35** marks.
- Calculators and mathematical formulæ booklet will be provided.

⚠ Attempt only **THREE** questions.

Question 1.

(a) Define the operation $\triangle$ on sets by $A \triangle B := (A \smallsetminus B) \cup (B \smallsetminus A)$.

(i) Construct a truth table showing when the statement "$x \in A \triangle B$" is true, depending on all the cases when "$x \in A$" and "$x \in B$" are true/false.

(ii) Draw a Venn Diagram for two sets A and B, highlighting the region corresponding to the set $A \triangle B$.

(iii) Use truth tables to prove the following statements (you may refer to your table from (i)).

(Γ) $A \triangle B = (A \cup B) \smallsetminus (A \cap B)$

(Ͱ) $(A \triangle B) \triangle (B \triangle C) = A \triangle C$

(ᛚ) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$

(b) We can formally phrase the pigeonhole principle in terms of functions:

$$\forall A, B \subseteq \mathbb{N}, \forall f : A \to B, (|A| > |B| \Rightarrow \exists x_1, x_2 \in A : f(x_1) = f(x_2) \wedge x_1 \neq x_2).$$

Express the statement above in idiomatic English, and give its negation as a formal statement.

(c) (i) Show that for any integer k, $k^2 + k$ is always even.

(ii) Show that if x is an odd integer, then there exists an integer y such that $x^2 = 8y + 1$.

(d) For $x, y \in \mathbb{Z}$, let $x \sim y$ if $x^2 - y^2$ is divisible by 4.

(i) Prove that $\sim$ is an equivalence relation.

(ii) Describe the equivalence classes of $\sim$.
How many distinct equivalence classes are there?

[16, 4, 8, 7 marks]

Question 2.

(a) You may assume that every rational number can be written in the form a/b where $a, b \in \mathbb{Z}$ and $\mathrm{hcf}(a, b) = 1$.

(i) Show using induction that every natural number n is either odd or even, i.e., we can express $n = 2k$ or $n = 2k + 1$ for some $k \in \mathbb{N}$.

(ii) For any integer x, show that if x^3 is even, then x is even.

(iii) Hence, show that $\sqrt[3]{2}$ is irrational.

(iv) Deduce that the number $\dfrac{2 + \sqrt[3]{2}}{3 - \sqrt[3]{2}}$ is irrational.

(b) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = (2x - 3y, x + 2y)$$

is a bijection, and find an expression for f^{-1}.

(c) Consider a "jumbled inequality" consisting of $n \geqslant 2$ blanks with inequality signs '$<$' and '$>$' interspersed, such as

$$\underline{\quad} < \underline{\quad} > \underline{\quad} > \underline{\quad} < \underline{\quad} > \underline{\quad} < \underline{\quad}\,.$$

The goal is to fill in the blanks so that each consecutive pair of numbers is properly related, e.g., $\underline{\ 1\ } < \underline{\ 6\ } > \underline{\ 5\ } > \underline{\ 4\ } < \underline{\ 7\ } > \underline{\ 2\ } < \underline{\ 3\ }$ in the case above.

Prove, by induction, that you can always fill in n blanks with the numbers $1, 2, \ldots, n$, using each of those numbers exactly once.

[*Hint for the inductive step: if the last sign is '$<$', then you can just use the IH for the first n spaces and add $n + 1$ on the end*.]

[15, 10, 10 marks]

Question 3.

(a) (i) State and prove the handshaking lemma.

(ii) Show that $\delta(G) \leqslant n \cdot \rho(G)$, where $\rho(G)$ denotes the *density* of the graph G, and $n = |V(G)|$.

(b) Consider the graph G depicted in figure 1. It is made up of four K_3's arranged in a cycle, such that the vertices of each K_3 are completely connected to the ones on either side.

(i) Explain why $\chi(G) \geqslant 6$.

[*Hint: look for a subgraph that you know requires 6 colours*.]

(ii) By constructing a 6-colouring of G, explain why $\chi(G) = 6$.

(iii) Draw the complement graph $\overline{G}$. What is $\chi(\overline{G})$?

(c) (i) Show that any tree has at least two leaves.

(ii) Prove that the number of edges in a tree on n vertices is $n-1$.

(iii) Show that a tree on n vertices whose degrees are all either 1 or 4 has precisely $\frac{2}{3}(n+1)$ leaves.

[*Hint: use the handshaking lemma*.]

(d) Suppose the graph G has no cycle of length smaller than 5. Show that $|V(G)| \geqslant \delta(G)^2 + 1$.

[*Hint: pick a vertex, look at its neighbours, and the neighbours of its neighbours*.]

[10, 7, 10, 8 marks]

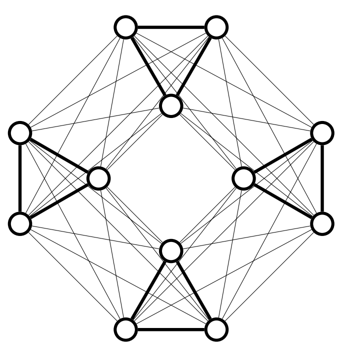


Figure 1: The graph G in 3(b)

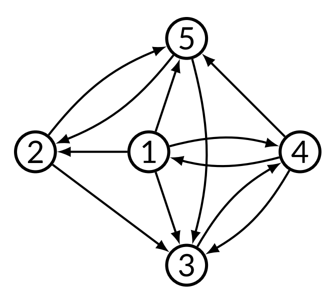


Figure 2: The network D in 4(c)

Question 4.

(a) Consider the two matrices $\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & -4 & 1 \\ -3 & 4 & 1 \\ 1 & -4 & 5 \end{pmatrix}$.

(i) Find **AB**, and deduce the matrix $\mathbf{A}^{-1}$.

(ii) Hence, solve the simultaneous equations

$$\begin{cases} 3x + 2y - z = 4 \\ 2x + 3y - z = 7 \\ x + 2y + z = 0. \end{cases}$$

(iii) Show that for any invertible matrix **M**, if λ is an eigenvalue of **M**, then $1/\lambda$ is an eigenvalue of $\mathbf{M}^{-1}$.

[*Hint: start with the definition of what an eigenvalue is:* $\mathbf{M}\boldsymbol{x} = \lambda\boldsymbol{x}$ *for some* $\boldsymbol{x} \neq \mathbf{0}$, *and introduce the matrix* $\mathbf{M}^{-1}$.]

(iv) Thus, given that $\det(\lambda\mathbf{I} - \mathbf{B}) = (\lambda - 2)(\lambda - 4)(\lambda - 8)$, deduce that the eigenvalues of **A** are 1, 2 and 4.

(v) Find an eigenvector for each eigenvalue of **A**.

(b) By constructing an appropriate diagram involving the basis vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, find the matrix which reflects vectors in $\mathbb{R}^2$ in the line $y = -x$.

(c) Consider the network of linked webpages represented by the digraph D in figure 2, together with the matrix

$$\mathbf{B} = 0.85 \begin{pmatrix} 0 & 0 & 0 & 1/3 & 0 \\ 1/4 & 0 & 0 & 0 & 1/2 \\ 1/4 & 1/2 & 0 & 1/3 & 1/2 \\ 1/4 & 0 & 1 & 0 & 0 \\ 1/4 & 1/2 & 0 & 1/3 & 0 \end{pmatrix} + 0.15 \begin{pmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{pmatrix}.$$

The vector $\boldsymbol{x} = (0.11, 0.14, 0.27, 0.29, 0.19)$ is an eigenvector for **B**, with corresponding eigenvalue $\lambda = 1$.

Explain in detail what the matrix **B** is in the context of Brin & Page's PageRank algorithm, and interpret the vector $\boldsymbol{x}$ as a ranking for the webpages.

[24, 4, 7 marks]

Answers and Hints

Note: The answers here should not be considered “model answers”, especially those involving proofs. The presentation here is terse, because the primary purpose of these solutions is to serve as a marking scheme for whoever is marking examination scripts.

1. (a) Note: This is just the symmetric difference of A and B.

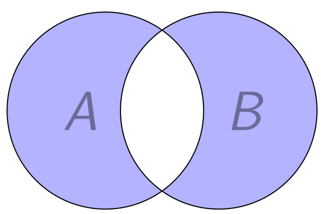
3 marks

(i) $A \triangle B$

$x \in A$	$x \in B$	$x \in (A \setminus B)$	$\cup$	$(B \setminus A)$
T	T	F	**F**	F
T	F	T	**T**	F
F	T	F	**T**	T
F	F	F	**F**	F

2 marks

(ii) The blue shaded region corresponds to $A \triangle B$:



3 marks

(iii) (Γ) $A \triangle B = (A \cup B) \setminus (A \cap B)$

We need to show that $x \in A \triangle B \leftrightarrow x \in (A \cup B) \setminus (A \cap B)$ is a tautology.*

$x \in A$	$x \in B$	$x \in (A \triangle B)$	$\leftrightarrow$	$x \in (A \cup B)$	$\setminus$	$(A \cap B)$
T	T	F	**T**	T	F	T
T	F	T	**T**	T	T	F
F	T	T	**T**	T	T	F
F	F	F	**T**	F	F	F

*Optionally, the student can translate this, using the definitions of $\triangle$, $\setminus$, $\cap$ and $\cup$, into the purely zeroth-order statement $(a \wedge \neg b) \vee (b \wedge \neg a) \leftrightarrow (a \vee b) \wedge \neg(a \wedge b)$, where $a \leftrightarrow (x \in A)$ and $b \leftrightarrow (x \in B)$, but this is not necessary, and the truth table columns will be identical.

4 marks

(ŀ) $(A \triangle B) \triangle (B \triangle C) = A \triangle C$

We need to show that $x \in (A \triangle B) \triangle (B \triangle C) \leftrightarrow A \triangle C$ is a tautology.

$x \in A$	$x \in B$	$x \in C$	$x \in (A \triangle B)$	$\triangle$	$(B \triangle C)$	$\leftrightarrow$	$x \in (A \triangle C)$
T	T	T	F	F	F	**T**	F
T	T	F	F	T	T	**T**	T
T	F	T	T	F	T	**T**	F
T	F	F	T	T	F	**T**	T
F	T	T	T	T	F	**T**	T
F	T	F	T	F	T	**T**	F
F	F	T	F	T	T	**T**	T
F	F	F	F	F	F	**T**	F

4 marks

(Ɩ) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$

We need to show that $x \in A \cap (B \triangle C) \leftrightarrow x \in (A \cap B) \triangle (A \cap C)$ is a tautology.

$x \in A$	$x \in B$	$x \in C$	$x \in A \cap$	$(B \triangle C)$	$\leftrightarrow$	$x \in (A \cap B)$	$\triangle$	$(A \cap C)$
T	T	T	F	F	**T**	T	F	T
T	T	F	T	T	**T**	T	T	F
T	F	T	T	T	**T**	F	T	T
T	F	F	F	F	**T**	F	F	F
F	T	T	F	F	**T**	F	F	F
F	T	F	F	T	**T**	F	F	F
F	F	T	F	T	**T**	F	F	F
F	F	F	F	F	**T**	F	F	F

2, 2 marks

(b) One way, in idiomatic English:

For any two subsets A and B of the natural numbers, and for any function going from A to B, if A is larger than B, then f sends two different points in A to the same point in B.†

The negation:

$$\exists A, B \subseteq \mathbb{N}, \exists f : A \to B : |A| > |B| \wedge (\forall x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2).$$

3 marks

(c) (i) This can be done by induction on k, or by case work (i.e., splitting into cases depending on whether k is odd or even), but the simplest it so argue directly, here is an outline of the proof: we have that $k^2 + k = k(k+1)$, and note that one of k or $k+1$ must be even, and therefore their product must be also. □

5 marks

(ii) Straightforward direct proof. Here is an outline: since x is odd, we can write it as $2k+1$, thus $x^2 = (2k+1)^2 = 4(k^2+k)+1$, and since k^2+k is even from part (i), we can say that $k^2+k = 2y$ for some $y \in \mathbb{Z}$, thus we have that $x^2 = 8y+1$. □

3 marks

(d) This is straightforward, we need to check reflexivity, symmetry and transitivity. An outline: for reflexivity, $x^2 - x^2 = 0$ which is divisible by 4; for symmetry we have that $y^2 - x^2 = -(x^2 - y^2)$, thus it is also divisible by 4, finally for transitivity we note that $x^2 - z^2 = (x^2 - y^2) + (y^2 - z^2)$, thus if both are divisible by 4 so is the former. □

4 marks

(e) Notice that for fixed x, we want to find all y such that for some k, $x^2 - y^2 = 4k$, i.e., $(x-y)(x+y) = 4k$. $x-y$ and $x+y$ are both even if x and y have the same parity,‡ and both odd otherwise. If $x+y$ and $x-y$ are both even, then their product is divisible by 4, and if they are both odd, then clearly the product is not divisible by 4 (it's not even even!).

Therefore we see that for any $x \in \mathbb{Z}$, we have the equivalence class $[x] = \{y \in \mathbb{Z} : y \text{ has the same parity as } x\}$, meaning that the relation $\sim$ partitions $\mathbb{Z}$ into two distinct equivalence classes: the odd numbers and the even numbers.

†i.e., f is *not injective*.

‡i.e., if x and y are both odd or both even.

3 marks

2. (a) (i) For the base case, $n = 1$ can be expressed as $2 \cdot 0 + 1$.

Now for the inductive step, suppose $n - 1$ can be expressed as $2k$ or $2k + 1$. In the first case, we have $n = 2k + 1$, and in the second case, $n = 2(k + 1)$, which completes the proof. □

4 marks

(ii) By contrapositive: we show that if x is not even, then x^3 is not even. By part (i), this is equivalent to: if x is odd, then x^3 is odd.

Therefore, suppose x is odd. Then we may write it as $x = 2k+1$ for some k. But then $x^3 = (2k + 1)^3 = 1 + 6k + 12k^2 + 8k^3 = 2(4k^3 + 6k^2 + 3k) + 1$, which is also clearly odd. □

4 marks

(iii) By contradiction: suppose $\sqrt[3]{2}$ is rational, so we may express $\sqrt[3]{2} = a/b$ with $\operatorname{hcf}(a, b) = 1$. But then $a^3 = 2b^3$, which means that a^3 is even, and so by part (ii) a is even, say $a = 2k$ for some k.

Then $a^3 = 2b^3 \implies (2k)^3 = 2b^3 \implies 2(2k^3) = b^3$, i.e., b^3 is even, which by (ii) implies that b is even.

Thus a and b are both divisible by 2. This contradicts that $\operatorname{hcf}(a, b) = 1$. □

4 marks

(iv) If the given number is rational, say equal to a/b with $a, b \in \mathbb{Z}$, then we may make $\sqrt[3]{2}$ subject of the equation to get that $\sqrt[3]{2} = (3a - 2b)/(a + b)$, contradicting that $\sqrt[3]{2}$ is irrational.

(b) We need to show that f is (i) functional, (ii) total, (iii) injective and (iv) surjective.

1 mark

(i) Clearly f is functional, since it unambiguously assigns a unique pair of coordinates to each input pair $(x, y) \in \mathbb{R}^2$.

1 mark

(ii) It is also clear that f is total, since it assigns every point in the domain $\mathbb{R}^2$ a corresponding pair of coordinates.

4 marks

(iii) To see that f is injective, suppose that $f(x,y) = f(a,b)$, i.e.,

$$(2x-3y, x+2y) = (2a-3b, a+2b)$$

$$\implies \quad \begin{cases} 2x-3y = 2a-3b & (1) \\ x+2y = a+2b & (2) \end{cases}$$

$$\implies \quad \begin{cases} 2x-3y = 2a-3b & (1) \\ 2x+4y = 2a+4b & 2\cdot(2) \end{cases}$$

and subtracting the first from the second equation gives us that $7y = 7b$, i.e., $y = b$, and then by (2) we clearly get that $x = a$. Thus, if $f(x,y) = f(a,b)$, then $(x,y) = (a,b)$, so f is injective.

4 marks

(iv) Finally, to see that f is surjective, take any point (x,y) in the codomain $\mathbb{R}^2$, and solve

$$f(a,b) = (x,y)$$

$$\implies \quad (2a-3b, a+2b) = (x,y)$$

$$\implies \quad \begin{cases} 2a-3b = x & (1) \\ a+2b = y & (2) \end{cases}$$

$$\implies \quad \begin{cases} 2a-3b = x & (1) \\ 2a+4b = 2y & 2\cdot(2) \end{cases}$$

subtracting gives $7b = 2y-x$, so $b = (2y-x)/7$ and then using (2), we get that $a = y - 2(2y-x)/7 = (2x+3y)/7$. Thus we see that

$$f\left(\tfrac{2x+3y}{7}, \tfrac{2y-x}{7}\right) = (x,y),$$

and since this works for all (x,y) in the codomain, we see that f is surjective.

We also immediately obtain a formula for f^{-1}, namely,

$$f^{-1}(x,y) = \left(\tfrac{2x+3y}{7}, \tfrac{2y-x}{7}\right). \quad \square$$

10 marks

(c) For the base case with $n = 1$, there is nothing to show.[§]

Now, let $n \geqslant 1$, and suppose that we have a jumbled inequality with $n+1$ gaps. We can insert the numbers $1, \dots, n$ in the first n places by the induction hypothesis. Now if the last symbol is $<$, we can just add $n+1$ on the end.

Otherwise, if it is $>$, we can increment each of the already placed numbers by 1, and naturally the ordering is still respected, so that now there are the numbers $2, \dots, n+1$ placed in the first n spaces, and finally we can place 1 at the end. $\square$

5 marks

3. (a) (i) For all graphs G,

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

Proof 1. The number $\deg(v)$ counts the number of edges incident to the vertex v. Since each edge in the graph is incident to precisely two vertices, then each edge in $|E(G)|$ contributes 2 to the sum. $\square$

Proof 2. We have

$$\begin{aligned} \sum_{v \in V(G)} \deg(v) &= \sum_{v \in V(G)} |N(v)| \\ &= \sum_{v \in V(G)} \sum_{e \in E(G)} \mathbb{1}_{v \in e} \\ &= \sum_{e \in E(G)} \sum_{v \in V(G)} \mathbb{1}_{v \in e} = \sum_{e \in E(G)} 2 = 2|E(G)|. \end{aligned}$$ $\square$

5 marks

(ii) First of all, observe that

$$(n-1)\rho(G) = \frac{(n-1)|E(G)|}{n(n-1)/2} = \frac{2|E(G)|}{|V(G)|}.$$

Now clearly

$$\begin{aligned} &\sum_{v \in V(G)} \delta(G) \leqslant \sum_{v \in V(G)} \deg(v) \\ &\implies |V(G)|\delta(G) \leqslant 2|E(G)|, \end{aligned}$$

[§] Optionally, the student can present the base case with $n = 2$, there are only two cases: _ < _ and _ > _, which are easily filled: $1 < 2$ and $2 > 1$.

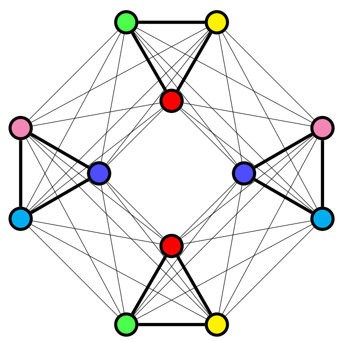
and dividing through by $|V(G)|$ gives

$$\delta(G) \leqslant \frac{2|V(G)|}{|E(G)|} = (n-1)\rho(G) \leqslant n \cdot \rho(G),$$

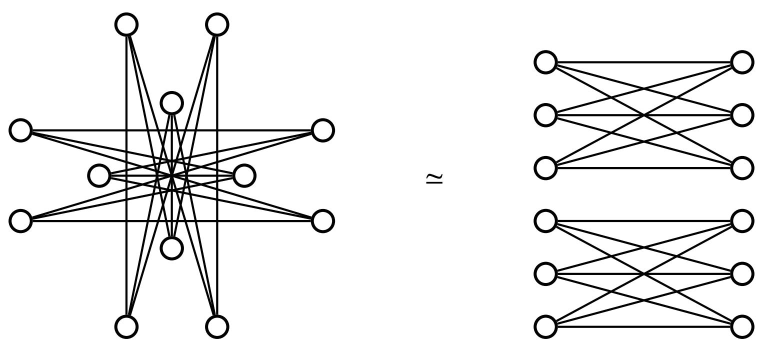
as required. □

3 marks (b) (i) If we take two adjacent K_3's in the cycle, we see that they induce K_6 as a subgraph. Thus $\chi(G) \geqslant \chi(K_6) = 6$.

2 marks (ii) Here's a possible 6-colouring:



2 marks (iii) The complement $\overline{G}$:



This graph is clearly bipartite, so $\chi(\overline{G}) = 2$.

3 marks (c) (i) Let P be a longest path in the tree. This necessarily has two leaves at its end, since otherwise it is not a longest path.

3 marks (ii) By induction on n. Clearly when $n = 1$ we have $0 = n-1$ edges, which establishes the base case. Now given a tree T on n vertices, remove a leaf ℓ (guaranteed to exist by (i)) to get $T - \ell$,

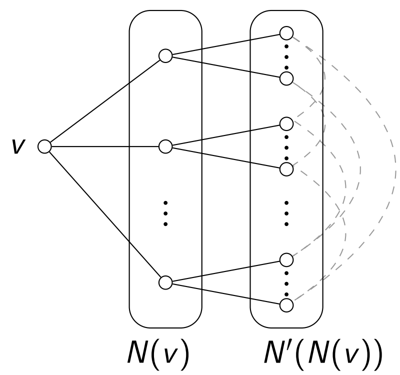


Figure a: Schematic representation of a vertex v, its neighbourhood $N(v)$ and $N'(N(v))$ in a graph with no cycles of length < 5: notice the vertices in $N'(N(v))$ are potentially allowed to be connected.

which by the IH has $(n-1)-1 = n-2$ edges. But adding ℓ back increases the number of edges by 1, so we have n–1 edges. □

4 marks

(iii) Suppose there are k vertices of degree 1. Then there are $n-k$ vertices of degree 4, and so the sum of degrees is $k+4(n-k)$, which by the handshaking lemma is $2|E(G)| = 2(n-1)$. Solving the equation $k+4(n-k) = 2(n-1)$ for k gives $k = \frac{2}{3}(n+1)$. □

7 marks

(d) Fix a vertex $v \in V(G)$, and consider the set of its neighbours, $N(v)$. No two vertices $x, y \in N(v)$ are connected, otherwise we'd get the cycle vxy of length 3 in the graph. There are at least $\delta(v)$ vertices in $N(v)$.

Each vertex $x \in N(v)$ has at least $\delta(G)-1$ neigbours outside of $\{v\}$ and $N(v)$ (we've said no two vertices in $N(v)$ can be connected to each other). Moreover, none of these can overlap, i.e., for any $x, y \in N(v)$, we have $N'(x) \cap N'(y) = \emptyset$ (where $N'(x)$ here is denoting $N(x) \setminus \{v\}$, because v is obviously included in the intersection but we don't want to count it again). Thus, each vertex $x \in N(v)$ gives rise to the existence of $\delta(G)-1$ distinct vertices in the graph.

Therefore we have

$$1 + |N(v)| + |N'(N(v))| = 1 + \delta + \delta(\delta - 1) = \delta(G)^2 + 1$$

vertices in the graph, as required. □

4 marks

4. (a) (i) The computation gives $\mathbf{AB} = 8\mathbf{I}$, so $\mathbf{A}^{-1} = \frac{1}{8}\mathbf{B}$.

4 marks

(ii) The given system is $\mathbf{A}\boldsymbol{x} = (4,7,0)$, so using $\mathbf{A}^{-1}$ from part (a), we have $\boldsymbol{x} = \mathbf{A}^{-1}(4,7,0) = \frac{1}{8}\mathbf{B}(4,7,0) = (-1,2,-3)$.

i.e., $x = -1, y = 2, z = -3$.

4 marks

(iii) If $\mathbf{M}\boldsymbol{x} = \lambda\boldsymbol{x}$, then $\mathbf{M}^{-1}\mathbf{M}\boldsymbol{x} = \mathbf{M}^{-1}(\lambda\boldsymbol{x})$, i.e., $\boldsymbol{x} = \lambda\mathbf{M}^{-1}\boldsymbol{x}$, so that $\mathbf{M}^{-1}\boldsymbol{x} = \frac{1}{\lambda}\boldsymbol{x}$. Thus $\frac{1}{\lambda}$ is an eigenvalue for $\mathbf{M}^{-1}$.

3 marks

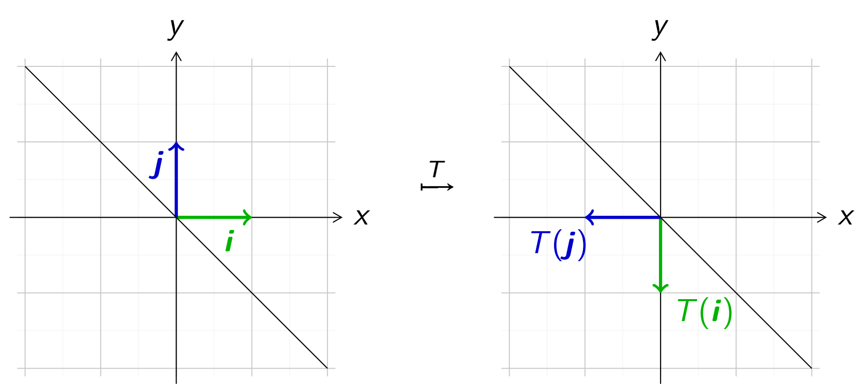
(iv) The eigenvalues of $\mathbf{B}$ are $2, 4$ and 8. The inverse of $\mathbf{B}$ is $\frac{1}{8}\mathbf{A}$. Thus $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{8}$ are eigenvalues of $\frac{1}{8}\mathbf{A}$, i.e., they satisfy $(\frac{1}{8}\mathbf{A})\boldsymbol{x} = \lambda\boldsymbol{x}$. But this means that $\mathbf{A}\boldsymbol{x} = 8\lambda\boldsymbol{x}$, i.e., the eigenvalues of $\mathbf{A}$ are 8λ for $\lambda = \frac{1}{2}, \frac{1}{4}$ and $\frac{1}{8}$, which gives $1, 2$ and 4.

9 marks

(v) Usual calculation, the eigenvector for $\lambda = 1$ is $(2,-1,2)$, the one for $\lambda = 2$ is $(1,1,3)$, and the one for $\lambda = 4$ is $(1,1,1)$.

4 marks

(b) Let's call this transformation T. Then:



Reading off the coordinates of $T(\boldsymbol{i})$ and $T(\boldsymbol{j})$, the desired matrix is easily seen to be

$$T = \begin{pmatrix} T(\boldsymbol{i}) & T(\boldsymbol{j}) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

7 marks

(c) We can view the internet as a digraph in which each directed edge indicates a link from one page to another. Brin & Page's PageRank algorithm takes the corresponding adjacency matrix $\mathbf{A}$, makes it stochastic by dividing each column by its outdegree, and then blends it with a small “teleportation” (uniform jump) term. This results in the matrix $\mathbf{B} = 0.85\mathbf{A} + 0.15\left(\frac{1}{n}\mathbf{J}\right)$, where $\mathbf{J}$ is the all-ones matrix, ensuring there are no zeroes (so we can apply the Perron–Frobenius theorem and ensure a unique stable-state).

The vector $\mathbf{x} = (0.11, 0.14, 0.27, 0.29, 0.19)$ is the principal eigenvector (eigenvalue 1) of $\mathbf{B}$. Interpreting these as probabilities, page 4, corresponding to 0.29 is ranked highest (about 29% of the time a “random surfer” would end up there), and the one corresponding to 0.11, page 1, is ranked lowest. Thus $\mathbf{x}$ acts as a steady-state distribution, measuring each page’s importance in this simple example.