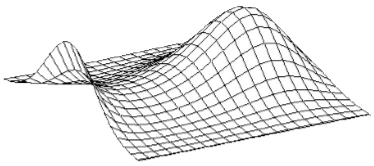


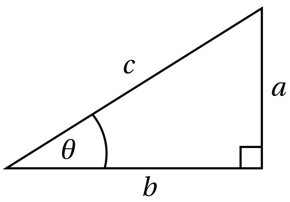
MATHEMATICS TUTORIALS
ĦAL TARXIEN

Trigonometry
Sine and Cosine Rules

MATSEC Intermediate Level **Notes and Past Paper Difficulties**

The Rules

In any right-angled triangle, we know that we can apply any of the following:

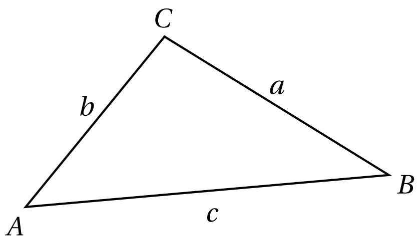


Pythagoras' Theorem
$a^2 + b^2 = c^2$

and

Trigonometric Ratios
$\sin\theta = \dfrac{a}{c} \qquad \cos\theta = \dfrac{b}{c} \qquad \tan\theta = \dfrac{a}{b}$

which we've used in countless situations. In fact, we sometimes split triangles which aren't right-angled into two right-angled ones, just so we can take advantage of the facts above. These facts, although very useful, still occasionally prove inadequate when dealing with triangles which aren't right-angled. We proceed to give two rules which can be used in any triangle, not just right-angled. Consider the triangle below:



We have labelled vertices in such a way so that the side of length a is opposite the angle at vertex A, and similarly for side b and vertex B, and side c and vertex C. We will give the two results below in terms of the above labelling.

The Sine Rule In any triangle, we have the following:

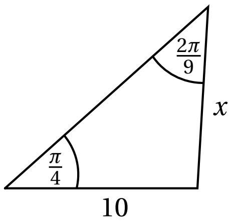
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This result is easy to memorise, and proves useful in the following cases:

i) finding the length of a side when given two angles and the length of any other side;

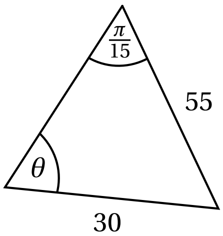
ii) finding an angle when given the length of two sides and an angle which is *not* between them.

Example Find the length x in the triangle below.



By the sine rule, $\dfrac{10}{\sin\frac{2\pi}{9}} = \dfrac{x}{\sin\frac{\pi}{4}} \implies 10\sin\frac{\pi}{4} = x\sin\frac{2\pi}{9} \implies x = \dfrac{10\sin\frac{\pi}{4}}{\sin\frac{2\pi}{9}} = 11$ units (to 2 d.p.s).

Example Find the angle θ in the triangle below.



By the sine rule, $\dfrac{30}{\sin\frac{\pi}{15}} = \dfrac{55}{\sin\theta} \implies 30\sin\theta = 55\sin\frac{\pi}{15} \implies \theta = \sin^{-1}\left(\dfrac{55\sin\frac{\pi}{15}}{30}\right) = 0.391$ (to 3 d.p.s).

The Cosine Rule In any triangle, we have the following:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

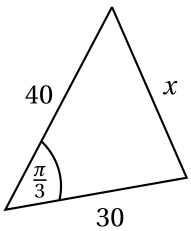
This result is not as easy as its predecessor to memorise, however it has a certain pattern to it which is easy to get used to. The analogous rules for different angles are $a^2 = b^2+c^2-2bc\cos A$ and $b^2 = a^2+c^2-2ac\cos B$. One may note that upon substituting the angle $\frac{\pi}{2}$, the result turns into none other than Pythagoras' theorem. Therefore we realise that the cosine rule is a more generalised form of the famous $a^2 + b^2 = c^2$ for any triangle.

The cosine rule is useful in the following scenarios:

i) finding an angle when given the lengths of all three sides;

ii) finding a side when given the length of two sides and the angle which *is* between them.

Example Find the length x in the triangle below.



By the cosine rule, $x^2 = 40^2 + 30^2 - 2(30)(40)\cos\frac{\pi}{3} \implies x^2 = 1300 \implies x = 10\sqrt{13}$ units.

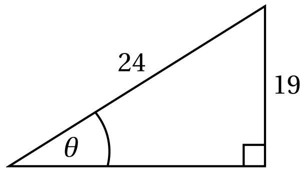
The proofs for both the sine and cosine rules are not in the scope of our syllabus, so they are not included in these notes. If you are curious about their derivation however, you may try proving them yourself, they're not that difficult and proving them yourself will give you a more intimate connection to them (*hint: split the triangles into two right-angled triangles, and combine Pythagoras' theorem, the trigonometric ratios and the fact that* $\sin^2\theta + \cos^2\theta \equiv 1$), or alternatively, take a look at their proofs here.

A Cool (Important) Trick

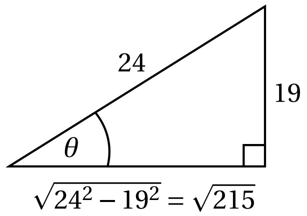
It's important that we realise exactly *what* a trigonometric ratio tells us about an angle. For example, if we know that

$$\sin\theta = \frac{19}{24},$$

what exactly does it mean? It's telling us that if we were to construct a *right-angled* triangle with θ as an angle, then the *ratio* of the opposite side to the hypotenuse will be 19:24, i.e. the information given by the equation $\sin\theta = \frac{19}{24}$ is equivalent to the information given by the diagram below:



and now by Pythagoras' theorem, we can determine what the other side is:



so now, using their basic definitions ($\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ and so on) we can determine that

$$\cos\theta = \frac{\sqrt{215}}{24}, \qquad \tan\theta = \frac{19}{\sqrt{215}}, \qquad \sec^2\theta = \frac{576}{215}, \qquad \text{and so on.}$$

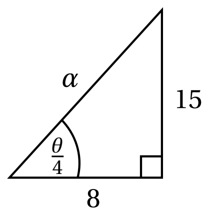
This is a very important observation: **if one trigonometric ratio is known, then all the others can also be determined**. This is a much more elegant and precise way to determine trigonometric ratios of angles, as opposed to doing the following:

$$\sin\theta = \frac{19}{24} \implies \theta = \sin^{-1}\frac{19}{24} = 0.9134321 \implies \cos\theta = 0.6109532$$

which is what a lot of students tend to do!

Example Given that $\cot\frac{\theta}{4} = \frac{8}{15}$, determine the other five trigonometric ratios.

$\cot\dfrac{\theta}{4} = \dfrac{8}{15} \implies \dfrac{1}{\tan\frac{\theta}{4}} = \dfrac{8}{15} \implies \tan\dfrac{\theta}{4} = \dfrac{15}{8}$, which graphically translates to:



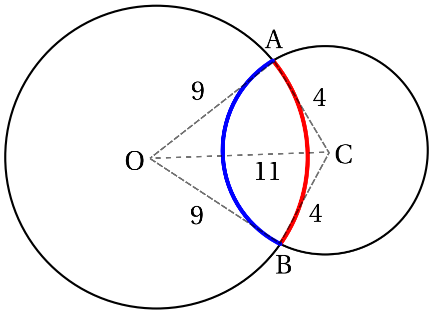
where $\alpha = \sqrt{15^2 + 8^2} = 17$, and therefore we can obtain the four remaining ratios:

$$\sin\frac{\theta}{4} = \frac{15}{17}, \qquad \cos\frac{\theta}{4} = \frac{8}{17}, \qquad \sec\frac{\theta}{4} = \frac{1}{\cos\frac{\theta}{4}} = \frac{17}{8}, \qquad \csc\frac{\theta}{4} = \frac{1}{\sin\frac{\theta}{4}} = \frac{15}{8}$$

Some Past Paper Questions

MATSEC May 2008 Q6(b) Two circles of radii 9cm and 4cm respectively are positioned so that their centres are 11cm apart. Find the perimeter which bounds the common region of the two circles.

We have the following:



We wish to find the perimeter of the red/blue region. This can simply be obtained by adding the length of the arc OAB (in red) to the length of the arc CAB (in blue). The length of the arc is given by the formula $L = r\theta$, where r is simply the radius of the circle and θ is the angle subtended at the centre, which we still need to determine for both circles.

Notice that the triangles OAC and OBC are symmetric in the line OC. This means that the angle we need for the arc OAB (in red), i.e. the angle $A\hat{O}B$, is twice the angle $A\hat{O}C$. Now in triangle AOC:

$$\begin{aligned}
4^2 &= 9^2 + 11^2 - 2(9)(11)\cos(A\hat{O}C) \\
\Rightarrow A\hat{O}C &= \cos^{-1}\left(\frac{9^2+11^2-4^2}{2(9)(11)}\right) = \cos^{-1}\frac{31}{33} \approx 0.3499 \\
\therefore A\hat{O}B &= 2 \times A\hat{O}C = 2\cos^{-1}\frac{31}{33} \approx 0.6999
\end{aligned}$$

as always, the result must be in radians for the formula to be applied. By a similar reasoning, we can obtain the angle $A\hat{C}B$, which is twice the angle $A\hat{C}O$:

$$\begin{aligned}
9^2 &= 4^2 + 11^2 - 2(4)(11)\cos(A\hat{C}O) \\
\Rightarrow A\hat{C}O &= \cos^{-1}\left(\frac{4^2+11^2-9^2}{2(4)(11)}\right) = \cos^{-1}\frac{7}{11} \approx 0.8810 \\
\therefore A\hat{C}B &= 2 \times A\hat{C}O = 2\cos^{-1}\frac{7}{11} \approx 1.7620
\end{aligned}$$

Hence the total perimeter is given by

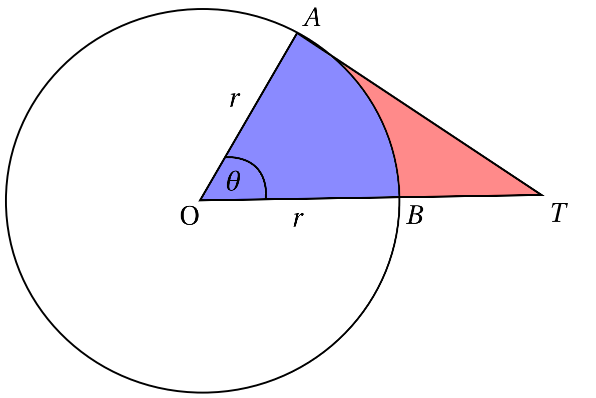
$$\begin{aligned}
L_{\text{total}} &= L_1 + L_2 \\
&= r_1\theta_1 + r_2\theta_2 \\
&= 9(A\hat{O}B) + 4(A\hat{C}B) \\
&= 18\cos^{-1}\frac{31}{33} + 8\cos^{-1}\frac{7}{11} \\
&\approx 13.3471 \text{ units.}
\end{aligned}$$

MATSEC Sept 2008 Q6(b) A circle has centre O and radius r. A and B are two points on its circumference so that angle $A\widehat{O}B$ is less than $\frac{\pi}{2}$. The tangent at A meets the radius OB, produced, at the point T. The area enclosed by AT, BT and the arc AB is equal to half the area of the sector AOB.

i) If the angle $A\widehat{O}B$ is denoted by θ, show that $3\theta = 2\tan\theta$.

ii) Working in radians, find by trial and error a value for the angle θ to the nearest 0.1 radian.

For part (i), we have the following scenario:



with the red area being half the size of the blue area; and we wish to show that $2\tan\theta = 3\theta$. The blue area is simply given by $\frac{1}{2}r^2\theta$. The red area can be determined by subtracting the blue area from the area of $\triangle OAT$. Hence we have

$$\begin{aligned}
\text{Area of } \triangle OAT - \text{Area of sector } OAB &= \frac{1}{2} \times \text{Area of sector } OAB \\
\frac{1}{2}(OA)(OT)\sin\theta - \frac{1}{2}r^2\theta &= \frac{1}{2}\left(\frac{1}{2}r^2\theta\right) \\
\Rightarrow \frac{1}{2}r(OT)\sin\theta - \frac{1}{2}r^2\theta &= \frac{1}{4}r^2\theta \\
\Rightarrow 2r(OT)\sin\theta - 2r^2\theta &= r^2\theta \\
\Rightarrow 2r(OT)\sin\theta &= 3r^2\theta \\
\therefore 2(OT)\sin\theta &= 3r\theta
\end{aligned}$$

Now we need to determine the unknown OT. Since AT is a tangent line, then $O\widehat{A}T$ is a right-angle. This means that

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{AT}{OT}$$

and also

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{AT}{OA} = \frac{AT}{r}$$

Now from the last equation, we can make AT subject, giving us $AT = r\tan\theta$. We can then substitute this in the previous equation giving us that

$$OT = \frac{r\tan\theta}{\sin\theta}$$

and finally we substitute this into the original equation:

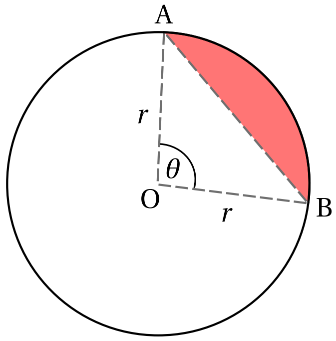
$$\begin{aligned}
\Rightarrow 2\left(\frac{r\tan\theta}{\sin\theta}\right)\sin\theta &= 3r\theta \\
\therefore 2\tan\theta &= 3\theta
\end{aligned}$$

For part (ii), trial and error gives $\theta \approx 0.97$.

MATSEC May 2009 Q6(b) A and B are two points on the circumference of a circle with centre O and radius r where angle $\mathrm{A\hat{O}B} = \theta$ radians ($\theta < \pi$). Prove that the area of the segment bounded by the chord AB and the minor arc is $\frac{1}{2}(\theta - \sin\theta)$.

A goat is tethered in a circular enclosure, centre O and radius r, by means of a rope of length r fixed to a point P on the circumference. Find, in terms of r, the area of the region that the goat can reach.

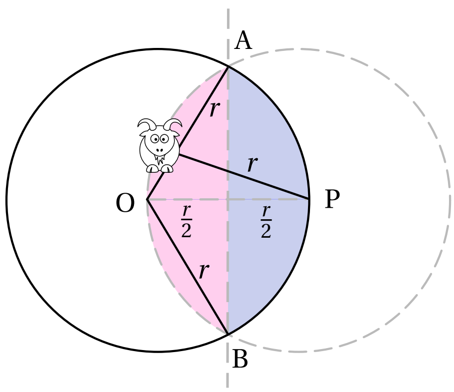
For the first part of this problem, we have the following:



where we want to determine the red area. Clearly this can be obtained by finding the area of the entire sector OAB, and subtracting the area of the triangle OAB, i.e.

$$\begin{aligned}\text{Total Area} &= \text{Area of Sector} - \text{Area of Triangle}\\ &= \frac{1}{2}r^2\theta - \frac{1}{2}(OA)(OB)\sin\theta\\ &= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta\\ &= \frac{1}{2}r^2(\theta - \sin\theta)\end{aligned}$$

Now for the second part, we have the following:



where we need to determine the red/blue area. We quickly see that the area is symmetric about the line AB, so if we determine half the area (say, the blue area only) then the entire area is equal to twice that area. Using the formula we derived in the first part of this question, we can determine the area given that we know the radius (which we do, it's simply our variable r) and the angle θ subtended at the centre, i.e. the angle at $\mathrm{A\hat{O}B}$. Now that angle is twice the angle $\mathrm{A\hat{O}P}$, which we can determine using trigonometric ratios in the right-angled triangle at the top left of the diagram (with hypotenuse r and adjacent side $\frac{r}{2}$). From this triangle, we can deduce that

$$\cos(\mathrm{A\hat{O}P}) = \frac{\frac{r}{2}}{r} = \frac{1}{2} \implies \mathrm{A\hat{O}P} = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$$

and therefore that angle $\mathrm{A\hat{O}B} = 2 \times \mathrm{A\hat{O}P} = \frac{2\pi}{3}$. Therefore by the formula we proved before, the blue area is equal to $\frac{1}{2}r^2\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right) = \frac{1}{2}r^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$, and hence the total red/blue area is twice that: $r^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ units^2.