Vectors

Pure Mathematics A-Level

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maths.com.mt/notes

We give a few basic facts about vectors, some strategies which are useful when solving some trivial problems involving lines and planes, and some mensuration formulæ.

Basics

- The *position vector* of a point A is the vector taking us from an origin O to the point, denoted $\vec{OA}$.

  In $\mathbb{R}^3$, the position vector for the point (a, b, c) is $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the vectors taking us from an origin to the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively.

- In general, a vector which takes us from a point A to B is denoted $\vec{AB}$, and is given by $\vec{OB} - \vec{OA}$.

- The *dot product* of two vectors $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, written $\mathbf{a}\cdot\mathbf{b}$, is a scalar defined as $a_1b_1 + a_2b_2 + a_3b_3$.

- The *magnitude* or *length* of a vector $\mathbf{v}$, written $\|\mathbf{v}\|$ (or sometimes just $|\mathbf{v}|$) is given by $\sqrt{\mathbf{v}\cdot\mathbf{v}}$.

  If a vector $\mathbf{v}$ has $\|\mathbf{v}\| = 1$, then $\mathbf{v}$ is said to be *unit*.

  In $\mathbb{R}^3$, the magnitude of the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by $\sqrt{a^2 + b^2 + c^2}$.

- The *direction* of a vector $\mathbf{v}$ is a unit vector which points in the direction of $\mathbf{v}$, denoted by $\hat{\mathbf{v}}$.

  In general, $\mathbf{v} = \|\mathbf{v}\|\hat{\mathbf{v}}$.

- Two vectors $\mathbf{a}$ and $\mathbf{b}$ are *parallel*, written $\mathbf{a} \parallel \mathbf{v}$, iff there exists some scalar $\lambda \in \mathbb{R}$ such that $\mathbf{a} = \lambda\mathbf{b}$.

  Equivalently, two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel iff $\hat{\mathbf{a}} = \pm\hat{\mathbf{b}}$.

- Two vectors $\mathbf{a}$ and $\mathbf{b}$ are *perpendicular*, written $\mathbf{a} \perp \mathbf{b}$, iff $\mathbf{a}\cdot\mathbf{b} = 0$.

- In general, $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

- A *vector equation* of a line[1] is an equation involving a general position vector $\mathbf{r}$. Similar to the equation of a line in $\mathbb{R}^2$, a point R with position vector $\mathbf{r}_0$ is on the line if and only if it satisfies the vector equation. The difference here is that a vector equation is not limited to two or three dimensions, but always represents a line in $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

- A *Cartesian equation* is an equation in terms of the variables x, y and z. Such equations may be used to represent lines[2], however they are limited to three dimensions or less.

Lines

- The vector equation of a line $\ell \subseteq \mathbb{R}^n$ is given by $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{a}$ is the position vector of any point A on ℓ, and $\mathbf{b}$ is a *direction vector* of ℓ, i.e. a vector pointing in the direction of the line.

  A point $A = (x, y, z) \in \mathbb{R}^3$ lies on the line ℓ if and only if there exists a value λ such that $\mathbf{r} = \vec{OA}$.

- The Cartesian equations of a line $\ell \subseteq \mathbb{R}^3$ through the point (x_1, y_1, z_1) with direction vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

- Given two lines $\ell_1 : \mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\ell_2 : \mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$, then
  - the *angle* between ℓ_1 and ℓ_2 is the angle between their directions, i.e. the angle between $\mathbf{b}_1$ and $\mathbf{b}_2$.
  - they are *parallel* if their directions are parallel, i.e. $\mathbf{b}_1 \parallel \mathbf{b}_2$.
  - they are *perpendicular* if their directions are perpendicular, i.e. if $\mathbf{b}_1 \cdot \mathbf{b}_2 = 0$.
  - they *intersect* if a pair of values λ and μ exist such that $\ell_1 = \ell_2$, i.e. $\mathbf{a}_1 + \lambda\mathbf{b}_1 = \mathbf{a}_2 + \mu\mathbf{b}_2$.
  - they are *skew* if they are neither intersecting nor parallel.

Some elementary problems

- *To find the equation of the line through two points A and B.*
  Let $\mathbf{a} = \vec{OA}$ and $\mathbf{b} = \vec{OB}$. Since these two points lie on ℓ, then the vector taking us from A to B, i.e. $\vec{AB} = \mathbf{b} - \mathbf{a}$, is pointing in the same direction as the line. Thus the line has equation $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$.

[1] or plane, or curve, or surface, etc.
[2] See [1] above.

- *To determine whether a pair of lines in $\mathbb{R}^3$ are intersecting, parallel or skew.*
  Let us denote the lines by $\ell_1 : \mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\ell_2 : \mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}_2$. First, we check if they are parallel, by checking whether $\hat{\mathbf{b}}_1 = \pm \hat{\mathbf{b}}_2$.

  If not, let us express the initial points as the vectors $\mathbf{a}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{a}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$, and the direction vectors as $\mathbf{b}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{b}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. If they intersect, then there are a pair of numbers λ, μ such that

$$\begin{aligned} \ell_1 &= \ell_2 \\ \implies \mathbf{a}_1 + \lambda \mathbf{b}_1 &= \mathbf{a}_2 + \mu \mathbf{b}_2 \\ \implies x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} + \lambda(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) &= x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k} \\ &\quad + \mu(a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) \\ \implies (x_1 + \lambda a_1)\mathbf{i} + (y_1 + \lambda b_1)\mathbf{j} + (z_1 + \lambda c_1)\mathbf{k} &= (x_2 + \mu a_2)\mathbf{i} \\ &\quad + (y_2 + \mu b_2)\mathbf{j} + (z_2 + \mu c_2)\mathbf{k}. \end{aligned}$$

  In other words, we want two values λ, μ which satisfy the 3 equations

$$\begin{cases} x_1 + \lambda a_1 = x_2 + \mu a_2 & (1) \\ y_1 + \lambda b_1 = y_2 + \mu b_2 & (2) \\ z_1 + \lambda c_1 = z_2 + \mu c_2 & (3) \end{cases}$$

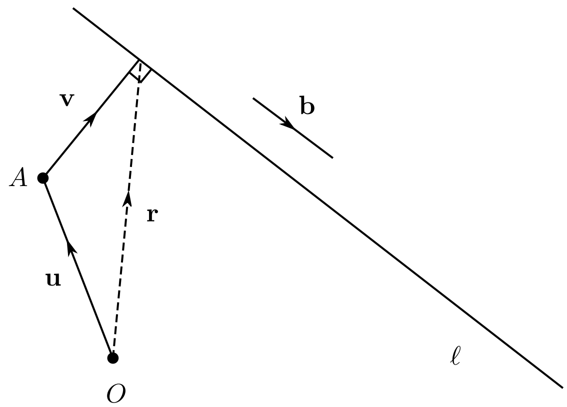
  Solve equations (1) and (2) simultaneously to obtain $\lambda = \lambda_0$, and $\mu = \mu_0$. Substitute these values into (3). If the equation is satisfied, then ℓ_1 and ℓ_2 intersect at the point $(x_1 + \lambda_0 a_1, y_1 + \lambda_0 b_1, z_1 + \lambda_0 c_1)$. Otherwise, ℓ_1 and ℓ_2 are skew.

- *To determine the angle between two lines.*
  Let the lines have direction vectors $\mathbf{b}_1$ and $\mathbf{b}_2$. Then the angle between the lines is the angle between their directions. Using the formula $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$, we get that

$$\cos\theta = \frac{\mathbf{b}_1 \cdot \mathbf{b}_2}{\|\mathbf{b}_1\|\|\mathbf{b}_2\|} = \frac{\mathbf{b}_1}{\|\mathbf{b}_1\|} \cdot \frac{\mathbf{b}_2}{\|\mathbf{b}_2\|} = \hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_2,$$

  therefore $\theta = \cos^{-1}(\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_2)$.

- *To find the distance between a line and a point A.*
  Let $\mathbf{u} = \vec{OA}$, let $\ell : \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ be the line in question, and let $\mathbf{v}$ be the vector taking us from the point A to the line ℓ. Clearly, we want to find the length of the vector $\mathbf{v}$:



From the diagram, $\mathbf{r} = \mathbf{u} + \mathbf{v}$, so $\mathbf{v} = \mathbf{r} - \mathbf{u}$, but $\mathbf{v}$ here depends on the value of λ which we substitute into the line. Clearly to get the shortest distance, we want that the vector $\mathbf{v}$ is perpendicular to the line ℓ, i.e. to be perpendicular to the direction vector $\mathbf{b}$. Thus we solve the equation $\mathbf{v} \cdot \mathbf{b} = 0$, i.e. $(\mathbf{r} - \mathbf{u}) \cdot \mathbf{b} = 0$, i.e. $(\mathbf{r} - \vec{OA}) \cdot \mathbf{b} = 0$.

This equation gives us a value of λ which, when substituted into the vector $\mathbf{v} = \mathbf{r} - \vec{OA}$, gives the shortest vector from the point A to the line ℓ. The required distance is then obtained simply by evaluating $\|\mathbf{v}\|$.

Planes

- The vector equation of a plane $\Pi \subseteq \mathbb{R}^n$ is given by $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, where $\mathbf{a}$ is the position of any point A of Π, and $\mathbf{n}$ is a *normal* of Π, i.e. a vector perpendicular to the plane.

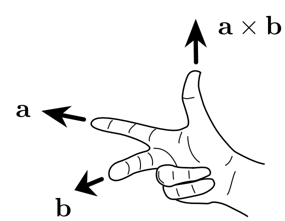
- The Cartesian equation of a plane $\Pi \subseteq \mathbb{R}^3$ with normal $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is $ax + by + cx = d$, where d is the result of $\mathbf{a} \cdot \mathbf{n}$ as in the vector equation.

- The *cross product* of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ in $\mathbb{R}^3$, written $\mathbf{a} \times \mathbf{b}$, is defined by the determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

The result is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, which points in the direction indicated by the right-hand rule.

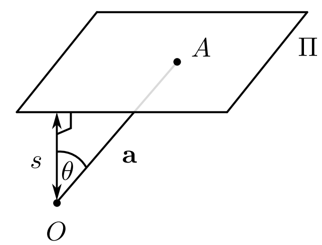
Note that $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$, since they point in opposite directions ($\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$), but $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta = \|\mathbf{b} \times \mathbf{a}\|$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.



Some elementary problems

- *Finding the distance of a plane from the origin.*
  Let $\Pi : \mathbf{r} \cdot \mathbf{n} = d$ be the plane in question, and let s be the required distance. The value d is $\mathbf{a} \cdot \mathbf{n}$, where $\mathbf{a}$ is the position vector of any point A on Π.

  Now $\mathbf{a} \cdot \mathbf{n} = \|\mathbf{a}\|\|\mathbf{n}\| \cos\theta = d$, where θ is the angle between the vector $\mathbf{a}$ and $\mathbf{n}$, i.e. the angle $\mathbf{a}$ makes perpendicular to the plane:

  From the diagram, $\cos\theta = \dfrac{s}{\|\mathbf{a}\|}$, thus $d = \|\mathbf{a}\|\|\mathbf{n}\|\dfrac{s}{\|\mathbf{a}\|} = \|\mathbf{n}\|s$.

  Therefore, the distance from the plane $\Pi : \mathbf{r} \cdot \mathbf{n} = d$ to the origin is given by $d/\|\mathbf{n}\|$.

- *Showing that a line is parallel to a plane.*
  If a line is parallel to a plane, then the direction of the line is perpendicular to the normal of the plane. Thus we show that $\mathbf{b} \perp \mathbf{n}$, i.e. that $\mathbf{b} \cdot \mathbf{n} = 0$.

- *Showing that a line is contained in a plane.*
  To show that a line $\ell : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ is contained in a plane Π, simply take any two distinct values of λ (say, $\lambda = 0$ and $\lambda = 1$) to give the position vectors of two particular points on the line. If we substitute these two positions for $\mathbf{r}$ into the equation for Π, and see that it is satisfied in both cases, then the line must lie in the plane Π.

- *Finding the point of intersection of a line and a plane.*
  If a line $\ell : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ intersects a plane $\Pi : \mathbf{r} \cdot \mathbf{n} = d$ at some point P, then the position vector of this point satisfies both of these vector equations. Thus we can substitute ℓ into Π, giving $(\mathbf{a} + \lambda\mathbf{b}) \cdot \mathbf{n} = d$, and solve this for the value of λ when the intersection occurs.

  The resulting λ, when substituted into the equation for ℓ, gives the position vector of the point P where ℓ and Π intersect each other.

- *Finding the angle between two planes.*
  The angle between two planes is simply the angle between their normals. Let these be $\mathbf{n}_1$ and $\mathbf{n}_2$. Using the formula $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$, we get

  $$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} = \frac{\mathbf{n}_1}{\|\mathbf{n}_1\|} \cdot \frac{\mathbf{n}_2}{\|\mathbf{n}_2\|} = \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2,$$

  therefore $\theta = \cos^{-1}(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2)$.

- *Finding the angle between a line and a plane.*
  Consider the angle ϕ between the direction vector $\mathbf{b}$ of the line ℓ and the normal $\mathbf{n}$ of the plane Π. Using the dot-product cosine formula, we know that ϕ is given by $\cos\phi = \hat{\mathbf{b}} \cdot \hat{\mathbf{n}}$. However observe that this angle is *not* the angle between the line and the plane, since the normal is perpendicular to the plane. Thus we need to correct this by $90°$, i.e. the angle we want is $\theta = 90° - \phi$.

  Thus $\cos\phi = \cos(90° - \theta) = \sin\theta = \hat{\mathbf{b}} \cdot \hat{\mathbf{n}}$. Therefore the acute angle between a line and a plane is given by

  $$\theta = \sin^{-1} |\hat{\mathbf{b}} \cdot \hat{\mathbf{n}}|.$$

- *Finding the distance between a point and a plane.*
  We have already seen that the distance from a plane to the origin is given by $s = d/\|\mathbf{n}\|$. One can show that the following more general result holds. The distance from the point A with position vector $\mathbf{a}$ to the plane $\Pi : \mathbf{r} \cdot \mathbf{n} = d$ is given by

  $$s = \left| \mathbf{a} \cdot \hat{\mathbf{n}} - \frac{d}{\|\mathbf{n}\|} \right|.$$

- *Finding the line of intersection of two planes.*
  Let $\Pi_1 : \mathbf{r} \cdot (a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) = d_1$ and $\Pi_2 : \mathbf{r} \cdot (a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) = d_2$ be the planes in question. These correspond to the Cartesian equations:

  $$\begin{cases} a_1x + b_1y + c_1z = d_1 & (1) \\ a_2x + b_2y + c_2z = d_2 & (2) \end{cases}$$

  Treating one of the variables as a constant (z, for example), we can solve these two simultaneous equations to obtain expressions for x and y in terms of z. From these equations, we can then solve for z, obtaining two equations of the form $z = \frac{x-x_1}{a}$ and $z = \frac{y-y_1}{b}$. These can be combined to give

  $$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - 0}{1},$$

  i.e. the Cartesian equations of a line with initial point $(x_1, y_1, 0)$ and direction vector $a\mathbf{i} + b\mathbf{j} + \mathbf{k}$. Thus the corresponding vector equation of the line is $\mathbf{r} = x_1\mathbf{i} + y_1\mathbf{j} + \lambda(a\mathbf{i} + b\mathbf{j} + \mathbf{k})$.

- *Finding the equation of a plane through three given points.*
  Let the three points be A, B and C. We first determine the vectors $\vec{AB}$ and $\vec{BC}$, two vectors which are contained in the plane. We can then take the cross product $\vec{AB} \times \vec{BC}$ to be the normal of the plane, since the resulting vector must perpendicular to both $\vec{AB}$ and $\vec{BC}$ (and therefore to the plane). Thus the desired plane is given by the vector equation

  $$\mathbf{r} \cdot (\vec{AB} \times \vec{BC}) = \vec{OA} \cdot (\vec{AB} \times \vec{BC}).$$

- *Finding the equation of a plane parallel to two lines, passing through a given point.*
  Suppose the two lines have directions $\mathbf{b}_1$ and $\mathbf{b}_2$, and the plane is to pass through the point A. Then by a similar reasoning to the previous problem, we take $\mathbf{b}_1 \times \mathbf{b}_2$ to be the normal, and therefore the plane has equation $\mathbf{r} \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = \vec{OA} \cdot (\mathbf{b}_1 \times \mathbf{b}_2)$.

- *Finding the intersection of three planes.*
  Express the equations of the three planes in Cartesian form:

$$\begin{cases} \Pi_1 : a_1 x + b_1 y + c_1 z = d_1 \\ \Pi_2 : a_2 x + b_2 y + c_2 z = d_2 \\ \Pi_3 : a_3 x + v_3 y + c_3 z = d_3 \end{cases}$$

  Reducing this system of equations (using Gaussian-elimination) gives the solution. Note that this can either be a point (unique solution), a line (infinitely many solutions) or non-existent (no solution):

  - If they intersect at a point, the solution will be the unique triple (x, y, z) given by solving the equations simultaneously.
  - If they intersect in a line, then Gaussian-elimination will reduce one of the rows to the equation $0x+0y+0z=0$, leaving two other planar equations. The line of intersection of the three initial planes is found by finding the line of intersection of these two (the solution to this is described previously).
  - If they do not intersect, Gaussian-elimination will reduce one of the rows to an equation with no solution ($0x+0y+0z=1$, for example).

- *Determining whether three vectors are linearly dependent/lie in a plane.*
  If the vectors $\mathbf{a}$ and $\mathbf{b}$ lie in a plane, then $\mathbf{a} \times \mathbf{b}$ is a normal to the plane. To determine whether a third vector $\mathbf{c}$ lies in the plane, observe that this is only the case when $\mathbf{c}$ is perpendicular to the normal $\mathbf{a} \times \mathbf{b}$, i.e. if $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = 0$. Note that this is the scalar-triple product of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, described in the next section.

Mensuration

- If the vectors $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$, then the *scalar-triple product* of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is the scalar $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$, denoted by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

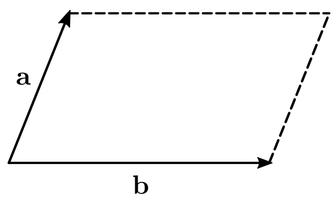
  This can be found evaluating the following determinant:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

  Note that interchanging $\cdot$ and $\times$ will not affect the result.

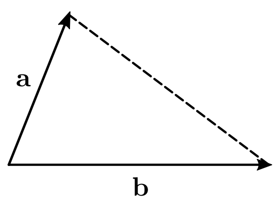
- The area of a *parallelogram* spanned by two vectors **a** and **b** is given by

$$A = \|\mathbf{a} \times \mathbf{b}\|.$$



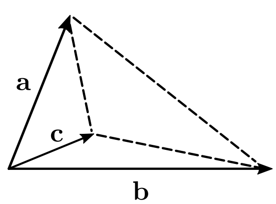
- The area of a *triangle* spanned by two vectors **a** and **b** is given by

$$A = \frac{1}{2}\|\mathbf{a} \times \mathbf{b}\|.$$



- The volume of a *tetrahedron* spanned by three vectors **a**, **b** and **c** is given by

$$V = \frac{1}{6}\left|[\mathbf{a}, \mathbf{b}, \mathbf{c}]\right|.$$



- The volume of a *parallelepiped* spanned by three vectors **a**, **b** and **c** is given by

$$V = \left|[\mathbf{a}, \mathbf{b}, \mathbf{c}]\right|.$$