Tutorial Sheet

Engineering Mathematics

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24th January, 2019

1 Questions

1. Determine, using Taylor's theorem, the power series expansion for the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sqrt{x^2 + 3},$$

centred at $x = 1$, up to the term in $(x-1)^4$.

2. The function $y(x)$ satisfies the differential equation

$$y' + xy + x^2 = 0$$

and the initial condition $y(1) = 0$. Approximate the y-coordinate at $x = 1.1$ using one iteration of the Runge–Kutta method.

3. Use the Leibniz–Maclaurin method to determine a power series solution to Chebyshev's differential equation

$$(1 - x^2)y'' + y = xy'$$

up to and including the term in x^6, given that $y(0) = 3$ and $y'(0) = -2$.

4. (a) Find the Laplace transforms $\mathcal{L}\{f\}$ and $\mathcal{L}\{g\}$ of the real-valued functions $f, g\colon [0, \infty) \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} t & \text{if } 0 \le t \le 3 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad g(t) = 3\cosh t - 8\cos 2t.$$

(b) Determine the following inverse Laplace transform, expressing it as a function of t:

$$\mathcal{L}^{-1}\left\{\frac{22+7s+4s^2}{(s^2+4)(s+5)}\right\}.$$

5. (a) Model by an ordinary differential equation the charge q in an RLC series circuit having a constant voltage source with inductance of 9 h, resistance of 12 Ω and capacitance of 0.25 F.

   (b) Solve the differential equation using Laplace transforms, given the initial conditions $q(0) = q_0$ and $\dot{q}(0) = 0$.

6. Legendre's equation is given by

$$(1-x^2)y'' - 2xy' + \xi(\xi+1)y = 0$$

where $\xi \in \mathbb{R}$ is a constant.

   (a) Determine the general solution in the special case when $\xi = 6$.

   (b) Hence deduce the Legendre polynomial $P_6(x)$.

2 Answers

1. $f(x) = 2 + \frac{x-1}{2} + \frac{3}{16}(x-1)^2 - \frac{3}{64}(x-1)^3 + \frac{3}{1024}(x-1)^4 + O\left((x-1)^5\right)$.
2. $y = -0.10482565$.
3. $y(x) = 3 - 2x - \frac{3x^2}{2} - \frac{3x^4}{8} - \frac{3x^6}{16} + O\left(x^7\right)$.
4. (a) Hint: for $\mathcal{L}f$, use the definition $\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt$.

   $\mathcal{L}\{f\}(s) = \frac{1}{s^2}(1 - (3s+1)e^{-3s})$.

   $$\mathcal{L}\{g\}(s) = \frac{5s(4-s^2)}{(s^2+4)(s^2-1)}.$$

   (b) $3e^{-5t} + \cos 2t + \sin 2t$.
5. (a) Hint: In general the equation is $L\ddot{q} + R\dot{q} + \frac{1}{C}\, q = \dot{V}$ where $V(t)$ is the voltage at time t. Since we have constant voltage source, then $V(t) = c \in \mathbb{R}$ so $\dot{V}(t) = 0$.

   Answer: $9\ddot{q} + 12\dot{q} + 4q = 0$.

   (b) $q(t) = \frac{q_0}{3} e^{-2t/3}(3 + 2t)$.
6. (a) Hint: Power series method (or Frobenius).

   $y(x) = a_0\left(1 - 21x^2 + 63x^2 - \frac{231}{5}x^6\right) + a_1\left(x - \frac{20}{3}x^3 + 10x^5 - \cdots\right)$.

   (b) Hint: The Legendre polynomial $P_6(x)$ is the linearly independent solution

   $$a_0\left(1 - 21x^2 + 63x^2 - \frac{231}{5}x^6\right)$$

   of part (a) which terminates, having a_0 chosen such that $P_6(1) = 1$.

   Take $a_0 = -\frac{5}{16}$, to get $P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$.