Probability Worksheet 2

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MMXIX

Notation

Sets (such as events) are denoted by capital letters such as A or X. In particular, sample spaces are denoted by the letters S or Ω. Calligraphic letters such as $\mathscr{A}$ or $\mathfrak{X}$ denote sets of sets, i.e., sets whose members are sets themselves. Small letters such as x or f denote elements of sets or functions. Sans-serif capital letters such as P or Q denote probability measures, i.e., countably additive functions on some probability space S such that $\mathsf{P}(S) = 1$.

The power set of a set A, i.e., the set of all subsets of A, is denoted by $\wp A$.

If $\mathfrak{X}$ is a family of sets, the union $\bigcup \mathfrak{X}$ is the set of all members of the members of $\mathfrak{X}$. For example, $\bigcup\{\{1,2\},\{1,3,5\},\{2,4\}\} = \{1,2,3,4,5\}$.

If $f\colon A \to B$ is a function from A to B, its range is the set $\{f(a) : a \in A\} \subseteq B$, denoted by $f(A)$.

Bayes' Theorem

1. (a) Let X be a set, and let $\mathscr{P} \subseteq \wp X$. When is $\mathscr{P}$ said to be a *partition* of X?

   (b) Let S be a probability space, and let $\mathscr{B}$ be an at most countable partition of S. Prove the *law of total probability*, i.e., for any event E,
   $$\mathsf{P}(E) = \sum_{B \in \mathscr{B}} \mathsf{P}(B)\,\mathsf{P}(E \mid B).$$

   (c) State and prove *Bayes' theorem.*

2. A group of students attend maths tutorials in Tarxien, and their tutor prepares tests for them on a regular basis. Being a fierce test-setter, the probability that a student passes a test is 45%.

   However, whether a student passes or not depends on how well-prepared they are. In fact, the probability that a student had studied for a test, given that he passed, is 96%. The problem is, that when a tutor announces a test, only 65% of the students study for it.

What is the probability of passing a test, given that the student has studied for it?

3. In the 2017 general election, the Labour Party (PL) got 54.8% of the vote, the Nationalist Party (PN) got 43.3%, and the rest of the votes were given to the smaller parties. The probability that someone who voted PL is from the south side of Malta is 0.6. The probability that someone who voted PN is from the south is 0.3, whereas the probability that someone who voted for a small party is from the south is 0.45.

   (a) What is the probability that someone from the south votes PN?

   (b) What is the probability that someone from the south votes for a small party?

   (c) What proportion of the voters are from the south?

4. Prove that $\mathsf{P}(\bar{A} \mid B) = 1 - \mathsf{P}(A \mid B)$.

5. John Larson wants to test his polygraph machine (lie detector). One hundred test subjects are told to lie, and the machine catches 80 of them in the lie. Another hundred test subjects are told to tell the truth, but the machine nevertheless thinks that 5 of them are lying.

   (a) Suppose a test subject is chosen at random from the 200, and the machine claims that he lied. What is the probability that he did in fact lie?

   (b) Suppose a test subject is chosen at random from the 200, and the machine claims that he told the truth. What is the probability that he did in fact tell the truth?

   (c) Let's say the police at Floriana begin to use the machine in interrogations, and suppose on average 15% of people arrested lie in their interrogations. When the machine indicates a lie, what is the probability that the suspect is really lying? If the machine does not indicate a lie, what is the probability that the suspect is really telling the truth?

   (d) Redo question (c) now assuming that only 3% of those arrested do lie.

6. A pub is supplied beer by three providers, A, B, and C. 30% comes from A, 20% from B and 50% from C. It is known that 2% of the beer from A is defective, 3% from B is defective, and 5% from C is defective.

   (a) If a beer is randomly selected from the pub, what is the probability that it is defective?

   (b) If a defective beer is found, what is the probability that it was supplied by B?

Solutions/Answers

1. (a) $\mathscr{P}$ is a partition of X when its members are pairwise disjoint ($P_1 \cap P_2 = \emptyset$ for each $P_1, P_2 \in \mathscr{P}$) and its union is X ($\bigcup \mathscr{P} = X$).

   (b) We have $E = E \cap S = E \cap (\bigcup_{B\in\mathscr{B}} B) = \bigcup_{B\in\mathscr{B}}(E \cap B)$, and since the sets $E \cap B$ are pairwise disjoint for different $B \in \mathscr{B}$, then $\mathsf{P}(E) = \mathsf{P}(\bigcup_{B\in\mathscr{B}}(E \cap B)) = \sum_{B\in\mathscr{B}} \mathsf{P}(E \cap B) = \sum_{B\in\mathscr{B}} \mathsf{P}(B)\,\mathsf{P}(E \mid B)$. $\square$

   (c) Bayes' theorem states that for any two events $A, B \subseteq S$,

$$\mathsf{P}(B \mid A) = \frac{\mathsf{P}(B)}{\mathsf{P}(A)}\ \mathsf{P}(A \mid B),$$

   or equivalently by part (b), if $\mathscr{B}$ is an (at most countable) partition of S and $B \in \mathscr{B}$, then the denominator can be written

$$\mathsf{P}(B \mid A) = \frac{\mathsf{P}(B)}{\sum_{B'\in\mathscr{B}} \mathsf{P}(B')\,\mathsf{P}(A \mid B')}\ \mathsf{P}(A \mid B).$$

   Proof. $\mathsf{P}(A)\,\mathsf{P}(B \mid A) = \mathsf{P}(A \cap B) = \mathsf{P}(B)\,\mathsf{P}(A \mid B)$. $\square$

2. Let P denote the event that a student passes a test, and let S denote the event that they studied. By Bayes' theorem,

$$\mathsf{P}(P \mid S) = \frac{\mathsf{P}(S \mid P)\,\mathsf{P}(P)}{\mathsf{P}(S)} = \frac{96\% \times 45\%}{65\%} = 66.45\%.$$

3. (a) 0.28 (b) 0.018 (c) 47%.

4. $\mathsf{P}(\bar{A} \mid B) + \mathsf{P}(A \mid B) = \frac{\mathsf{P}(\bar{A}\cap B)+\mathsf{P}(A\cap B)}{\mathsf{P}(B)} = \frac{\mathsf{P}[(\bar{A}\cap B)\cup(A\cap B)]}{\mathsf{P}(B)} = \frac{\mathsf{P}(B)}{\mathsf{P}(B)} = 1.$ $\square$

5. (a) 0.94 (b) 0.83 (c) 0.74, 0.96 (d) 0.99, 0.33

6. (a) 0.037 (b) 0.16

Random Variables

1. (a) Define a random variable. What is the difference between a discrete and continuous random variable?

   (b) Define the probability mass function $\mathsf{P}(X = x)$ of a random variable X.

   (c) Define the cumulative distribution function $\mathsf{P}(X \leqslant x)$ of a random variable X.

2. Suppose a pair of dice are tossed, together with a coin. If the coin shows up heads, then the two numbers are added together, otherwise, the larger of the two numbers is taken.

(a) Describe an appropriate probability space and random variable to model this process.

(b) Express $\mathsf{P}(X = x)$ and $\mathsf{P}(X \leqslant x)$ in a concise way.

(c) Plot the probability mass function and the cumulative distribution function for this random variable.

Solutions/Answers

1. (a) In a probability space S, a random variable is a function $X: S \to \mathbb{R}$. A discrete random variable has countable range $X(S)$, whereas a continuous random variable has uncountable range.

   (b) The probability mass function of a random variable X is the function $\mathsf{P}(X = \cdot\,): X(S) \to [0,1]$ defined by

   $$\mathsf{P}(X = x) = \mathsf{P}(\{s \in S : X(s) = x\}).$$

   (c) The cumulative distribution function of a random variable X is the function $\mathsf{P}(X \leqslant \cdot\,): \mathbb{R} \to [0,1]$ defined on all real numbers by

   $$\mathsf{P}(X \leqslant x) = \mathsf{P}(\{s \in S : X(s) \leqslant x\}).$$

   If X is a discrete random variable, this is equivalent to the sum $\sum_{i \leqslant x} \mathsf{P}(X = i)$.

2. (a) The probability space is $S = \{1,2,3,4,5,6\}^2 \times \{H,T\}$, where is elementary event has equal probability ($\mathsf{P}(\{s\}) = 1/72$ for each $s \in S$). The corresponding random variable is $X: S \to \mathbb{R}$ defined by

   $$X(a,b,c) = \begin{cases} a+b & \text{if } c = H \\ \max\{a,b\} & \text{otherwise.} \end{cases}$$

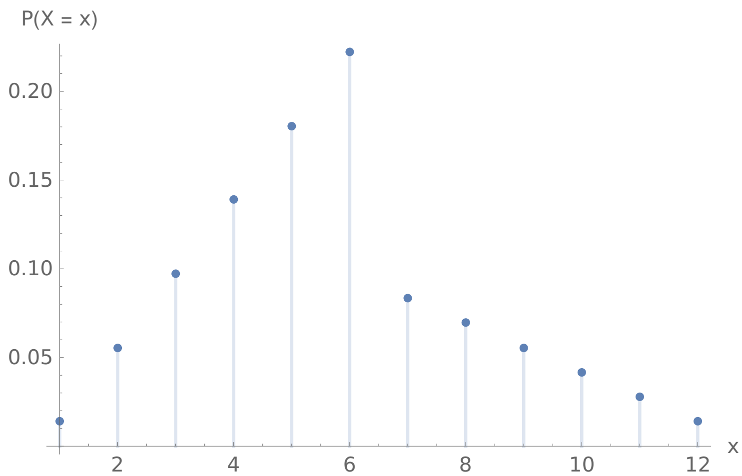
   (b) The probability density function $\mathsf{P}(X = \cdot\,): X(S) \to [0,1]$ is given by

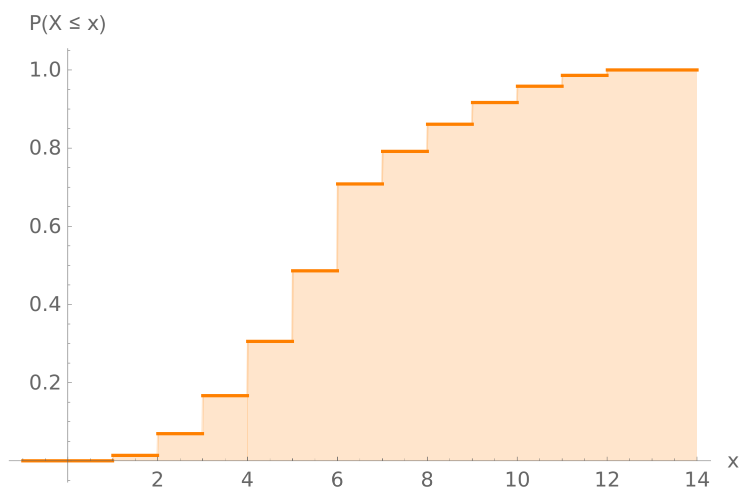
   $$\mathsf{P}(X = x) = \begin{cases} \frac{3x-2}{72} & \text{if } x \in \{1,\dots,6\} \\ \frac{13-x}{72} & \text{if } x \in \{7,\dots,12\}, \end{cases}$$

   and the cumulative density function $\mathsf{P}(X \leqslant \cdot\,): \mathbb{R} \to [0,1]$ is given by

   $$\mathsf{P}(X \leqslant x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{\lfloor x \rfloor}{144}(3\lfloor x \rfloor - 1) & \text{if } 1 \leqslant x < 7 \\ \frac{\lfloor x \rfloor}{144}(25 - \lfloor x \rfloor) - \frac{1}{12} & \text{if } 7 \leqslant x \leqslant 12 \\ 1 & \text{if } 12 \leqslant x. \end{cases}$$

   (c) PTO for plots.


P(X = x)
0.20
0.15
0.10
0.05
2
4
6
8
10
12
x


P(X ≤ x)
1.0
0.8
0.6
0.4
0.2
2
4
6
8
10
12
14
x