SOME EXERCISES IN PROPOSITIONAL LOGIC

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You may assume the following inference rules for propositional logic.

Truth and Falsehood

$$\frac{}{\top}\ \top\text{-INT} \qquad\qquad \frac{\bot}{A}\ \bot\text{-ELIM}$$

Conjunction

$$\frac{A\,,\,B}{A \wedge B}\ \wedge\text{-INT} \qquad \frac{A \wedge B}{A}\ \wedge\text{-ELIM}_1 \qquad \frac{A \wedge B}{B}\ \wedge\text{-ELIM}_2$$

Disjunction

$$\frac{A}{A \vee B}\ \vee\text{-INT}_1 \qquad \frac{B}{A \vee B}\ \vee\text{-INT}_2 \qquad \frac{A \to C\,,\,B \to C\,,\,A \vee B}{C}\ \vee\text{-ELIM}$$

Implication

$$\frac{A \to B\,,\,A}{B}\ \to\text{-ELIM} \qquad\qquad \frac{\Gamma\,,\,A \vdash B}{\Gamma \vdash A \to B}\ \to\text{-INT}$$

Biconditional

$$\frac{A \to B\,,\,B \to A}{A \leftrightarrow B}\ \leftrightarrow\text{-INT} \qquad \frac{A \leftrightarrow B}{A \to B}\ \leftrightarrow\text{-ELIM}_1 \qquad \frac{A \leftrightarrow B}{B \to A}\ \leftrightarrow\text{-ELIM}_2$$

Negation

$$\frac{\neg\neg A}{A}\ \neg\text{-ELIM} \qquad\qquad \frac{A \to B\,,\,A \to \neg B}{\neg A}\ \neg\text{-INT}$$

I. EXERCISES

1. Construct a Fitch proof for each of the following.

   (a) $\vdash A \leftrightarrow A$

   (b) $p \vee q\,,\, \neg p \vdash q$

   (c) $\neg\varphi \vee \psi \vdash \neg(\varphi \wedge \neg\psi)$

   (d) $P \vee Q\,,\, R \vee S\,,\, \neg Q \wedge \neg R \vdash P \wedge S$

   (e) $P \vee (Q \wedge R) \vdash (P \vee Q) \wedge (P \vee R)$

   (f) $\vdash (P \vee \neg Q) \vee (\neg P \vee Q)$

   (g) $P \vee \bot \vdash P$

   (h) $(a \wedge \neg b) \vee (\neg a \wedge b) \vdash (a \leftrightarrow b)$

   (i) $P \wedge \neg P \vdash Q$

2. Explain the difference between implication ($\to$) and semantic entailment ($\vdash$). Hence, explain in words what the rule $\to$-INT is saying.

⋆ 3. *Epistemic logic* is an extension of propositional logic by the pair of unary connectives $\Box$ (necessity) and $\Diamond$ (possibility). We use this logic to talk about what a person X knows. Intuitively, $\Box p$ means that X is convinced that p is true, whereas $\Diamond p$ means that X believes p is possible.

   Their behaviour is determined by these rules.

$$\frac{\Box(A \to B)}{\Box A \to \Box B}\,\mathbf{K} \qquad \frac{\Box A}{\Box\Box A}\,\mathbf{4} \qquad \frac{\neg\Box A}{\Box\neg\Box A}\,\mathbf{5}$$

$$\frac{A}{\Box\Diamond A}\,\mathbf{B} \qquad \frac{\Box A}{\Diamond A}\,\mathbf{D} \qquad \frac{\Box A}{\neg\Diamond\neg A}\,\textsc{dual}_1 \qquad \frac{\Diamond A}{\neg\Box\neg A}\,\textsc{dual}_2$$

   Notice that just because X knows something, doesn't mean that is true. (In other words, $\Box A \vdash A$ is not a rule.)

   (a) Translate each of the rules **K**, **4**, **5**, **B**, **D**, $\textsc{dual}_1$ and $\textsc{dual}_2$ into English so that you understand them better. For instance, $\textsc{dual}_1$ becomes

   > *If X thinks that A is true, then he doesn't believe that A is not possible.*

(b) Construct a Fitch proof for the following.

i. $P \vdash \Diamond\Diamond P$

ii. $\Box P, \Box(P \to Q) \vdash \Box Q$ (Modus Ponens)

iii. $\Box(P \to Q) \vdash \Diamond P \to \Diamond Q$

Now adding the rule $\frac{A}{\Box A}$ **G**,

iv. $\Box A \wedge \Box B \vdash \Box(A \wedge B)$

II. SOLUTIONS

Note that Finch proofs can be constructed in many different ways, so if your solution doesn't match exactly with one presented here, it doesn't mean that yours is incorrect.

1. (a) $\vdash A \leftrightarrow A$

1	$\quad A$	(subhypothesis)
2	$\quad A$	(line 1)
3	$A \to A$	($\to$-INT, 1–2)
4	$A \leftrightarrow A$	($\leftrightarrow$-INT, 3, 3)

High-level idea: We want to use $\leftrightarrow$-INT, so we're going to need an implication $A \to A$. But this is easily obtained by taking A as a hypothesis and applying $\to$-INT.

(b) $p \vee q, \neg q \vdash q$

1	$p \vee q$	(hypothesis)
2	$\neg p$	(hypothesis)
3	$\quad p$	(subhypothesis)
4	$\quad\quad \neg q$	(subsubhypothesis)
5	$\quad\quad \neg p$	(line 2)
6	$\quad \neg q \to \neg p$	($\to$-INT, 4–5)
7	$\quad\quad \neg q$	(subsubhypothesis)
8	$\quad\quad p$	(line 3)
9	$\quad \neg q \to p$	($\to$-INT, 7–8)
10	$\quad \neg\neg q$	($\neg$-INT, 9, 6)
11	$\quad q$	($\neg$-ELIM, 10)
12	$p \to q$	($\to$-INT, 3–11)
13	$\quad q$	(subhypothesis)
14	$\quad q$	(line 13)
15	$q \to q$	($\to$-INT, 13–14)
16	q	($\vee$-ELIM, 12, 15, 1)

High-level idea: We want to use $\vee$-ELIM to end up with a solitary q. Since we have $p \vee q$, it might be simplest to use this in $\vee$-ELIM, then we just need to prove $p \to q$ and $q \to q$.

Obtaining the latter is straightforward (take q as a subhypothesis, copy it on the next line and apply $\to$-INT). For the former, we know that $\neg p$ is true since it's one of our hypotheses, so intuitively, assuming p should allow us to deduce anything (including q). We can show that $\neg q$ implies both p and $\neg p$ (just by copying them from their respective line numbers), this will lead to $\neg\neg q$ by $\neg$-INT. Removing the double negation with $\neg$-ELIM will complete the proof that $p \to q$.