I. INTRODUCTION

THE word *algebra*, from the Arabic *al-jebr* (in Maltese, *għall-ġabra*) refers to the study of manipulation of mathematical symbols. The goal of algebra is to abstract away unnecessary details so that we can think more generally, using symbols in place of more concrete objects (such as numbers).

Throughout these notes, we assume basic symbols and notations of sets. Take a look at appendix A if any of the symbols are new to you.

We take quite a formal approach to things, building off definitions and proving things every step of the way. It is not important to learn the proofs here by heart, but it is important to read and understand them well. In the exam, you will have to produce proofs of your own, often of results which you have never seen before. Thus the way proofs are written and presented, as well as the difference between what we are allowed to assume and what we aren't, are essential to proving things correctly.

Moreover, in more advanced topics, mathematical jargon of the kind we present here is vital to be able to explain things as succinctly as possible. Thus, even if it seems like we are using verbose language to describe something which could otherwise be stated more matter-of-factly; the goal is to familiarise you with language we will rely on in the future.

It is important to work through all the exercises, not only to reinforce what you have learned, but to also garner sufficient instincts for what is to come. It is not enough to be able to do the exercises—by the end of them, you should be able to do similar exercises *easily*, almost without thinking. This way, when we go on to more advanced topics, your focus will be entirely on the new material, and you will not sacrifice any of your brain's "processing power" to understand the basic algebra.

When exercises are annotated with a ☕ symbol, this is instructing you to pour yourself some tea and dedicate some time to think about the problem, it might be harder than the others.

PRELIMINARY TECHNIQUES

Before we start the material of the course, you are encouraged to work through the following exercise. The ability to solve such basic problems will be assumed.

Exercise 1.1. 1. LINEAR EQUATIONS IN ONE VARIABLE

Solve the following equations.

a) $3x+4=16$

b) $7x-4=24$

c) $3x+4=x+16$

d) $3x-5+2x=1-x+5x$

e) $\frac{x}{2}-\frac{x}{3}=8$

f) $\frac{y}{6}+\frac{y}{4}=5$

g) $\frac{n}{5}+n=\frac{n}{3}+13$

h) $\frac{t}{4}-\frac{t}{5}=\frac{t}{2}-18$

i) $\frac{m}{3}+\frac{m}{2}=m-\frac{1}{6}$

j) $\frac{\theta}{4}+\frac{1}{2}+3\theta=2\theta+3$

k) $\frac{5}{q+5}=\frac{3}{q+7}$

l) $\frac{1-y}{1+y}=\frac{2}{3}$

m) $\frac{1}{7}(x+1)-\frac{1}{11}(x-2)=1$

n) $\frac{1}{x+1}=\frac{3}{4(x+1)}+\frac{1}{12}$

o) $\frac{4}{2s+1}-\frac{2}{3(2s+1)}=\frac{5}{9}$

p) $\frac{1}{7}(x+2)+3=\frac{x+3}{2}$

q) $\frac{3x}{2}\left(2+\frac{2}{x}\right)+7x-3=\frac{1}{4}(39x+7)$

2. SIMULTANEOUS LINEAR EQUATIONS IN TWO VARIABLES

Solve each of the following systems of equations.

a) $\begin{cases} 6x+2y=30 \\ 4x+3y=30 \end{cases}$

b) $\begin{cases} 2x+21=5y \\ 4x+3y=23 \end{cases}$

c) $\begin{cases} 3x+33=9y \\ 5x+4y=40 \end{cases}$

d) $\begin{cases} 3x+3=6y \\ 5x-6y=7 \end{cases}$

e) $\begin{cases} x+9y=34 \\ 4x-5y=13 \end{cases}$

f) $\begin{cases} 6x-3y=3 \\ 4x+5=3y \end{cases}$

g) Alice has more money than Bob. If Alice gave Bob €20, they would have the same amount. While if Bob gave Alice €22, Alice would then have twice as much as Bob. How much does each one actually have?

h) A woman is now 30 years older than her son. 15 years ago, she was twice as old. What are the present ages of the woman and her son?

3. LINEAR INEQUALITIES IN ONE VARIABLE

    a) $x+1>2$
    b) $5-3x<17-2x$
    c) $5(1+x)-2(1-x)\geqslant 2+3x$
    d) $\dfrac{x+3}{2}\leqslant\dfrac{x-2}{3}$

4. Solve for x.

    a) $x+y=2x-3z$
    b) $x+yx+zx=3-x+4xyz$
    c) $x=\dfrac{x+y+z^2}{y-3z}$
    d) $\dfrac{1}{x}+\dfrac{1}{w}=\dfrac{1}{z}$
    e) $\dfrac{1}{w}+\dfrac{1}{y}+\dfrac{1}{x}=\dfrac{1}{z}$
    f) $yz=\dfrac{y+{}^z/_x}{z+{}^y/_x}$

5. A man was looking at a portrait. Someone asks him: "Whose picture are you looking at?" He replied: "Brothers and sisters I have none, but this man's father, is my father's son."

    Whose picture was the man looking at?

6. Consider three brothers named John, James and William. John and James (the two J's) always lie, but William always tells the truth. The three are indistinguishable in appearance. You meet one of the three brothers on the street one day and wish to find out whether he is John (because John owes you money). You are allowed to ask him one question answerable by yes or no, but the question may not contain more than three words!

    What question would you ask?[a]

[a] This delightful problem was taken from the excellent book *To Mock a Mocking Bird* by Raymond Smullyan, which provides a nice informal introduction to combinatory logic using these kinds of logical puzzles (equivalent to the λ-calculus, for those of who study computing.)

THE REAL NUMBERS

The set of real numbers $\mathbb{R}$ is the object we work with the most throughout the course, although we will consider other mathematical objects in later topics (functions, vectors, matrices, complex numbers, etc.). The essential, defining properties which the real numbers obey are summarised nicely in theorem 1.2.

Theorem 1.2 (ℝ is an Ordered Field)**.** *Let $x, y, z \in \mathbb{R}$, and let $+$ and $\cdot$ denote the operations of addition and multiplication. Then the following properties hold.*

I. $x + y \in \mathbb{R}$ (CLOSURE OF +)

II. $x + (y + z) = (x + y) + z$ (ASSOCIATIVITY OF +)

III. *There is a number* $0 \in \mathbb{R}$ *such that for any* x,
$x + 0 = 0 + x = x$ (IDENTITY FOR +)

IV. *For each* x, *there is a number* $-x \in \mathbb{R}$, *such that*
$x + -x = -x + x = 0$ (INVERSE FOR +)

V. $x + y = y + x$ (COMMUTATIVITY OF +)

VI. $x \cdot y \in \mathbb{R}$ (CLOSURE OF ·)

VII. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (ASSOCIATIVITY OF ·)

VIII. *There is a number* $1 \in \mathbb{R}$ *such that* $1 \neq 0$, *and for any* x,
$x \cdot 1 = 1 \cdot x = x$ (IDENTITY FOR ·)

IX. *For each* x, *if* $x \neq 0$, *then there is a number* x^{-1} *such that*
$x \cdot x^{-1} = x^{-1} \cdot x = 1$ (INVERSE FOR ·)

X. $x \cdot y = y \cdot x$ (COMMUTATIVITY OF ·)

XI. $x \cdot (y + z) = x \cdot y + x \cdot z$ (DISTRIBUTIVITY OF · OVER +)

XII. *If* $x \leqslant y$ *and* $y \leqslant x$, *then* $x = y$ (ANTISYMMETRY OF ⩽)

XIII. *If* $x \leqslant y$ *and* $y \leqslant z$, *then* $x \leqslant z$ (TRANSITIVITY OF ⩽)

XIV. $x \leqslant y$ *or* $y \leqslant x$ (TOTALITY OF ⩽)

XV. *If* $x \leqslant y$, *then* $x + z \leqslant y + z$ (ORDER PRESERVATION +)

XVI. *If* $0 \leqslant x$ *and* $0 \leqslant y$, *then* $0 \leqslant x \cdot y$ (ORDER PRESERVATION ·)

We are actually glossing over a key detail here. Upon reading these "defining properties", a natural question you might ask is: "then what's the difference between ℚ and ℝ?", and indeed, you would be right in asking this, because the rationals are also an ordered field (i.e., they obey all the properties I–XVI of theorem 1.2 too). The rationals ℚ are, in a sense, the "smallest" set of numbers obeying these properties, but it has "holes", i.e., numbers which "should" be there, but aren't (such as $\sqrt{2}$, which we proved is not rational in appendix A). When we think of the number line, we think of a continuum of numbers without holes. So in the case of $\sqrt{2}$, we have the approximations

$$\begin{aligned}
1.4 &= \frac{14}{10} & 1.4^2 &= 1.96\\
1.41 &= \frac{141}{100} & 1.41^2 &= 1.9881\\
1.414 &= \frac{1\,414}{1\,000} & 1.414^2 &= 1.999396\\
1.4142 &= \frac{14\,142}{10\,000} & 1.4142^2 &= 1.99996164\\
1.41421 &= \frac{141\,421}{100\,000} & 1.41421^2 &= 1.9999899241\\
1.414213 &= \frac{1\,414\,213}{1\,000\,000} & 1.414213^2 &= 1.999998409369\\
&\vdots
\end{aligned}$$

each of which is rational, but we should also have a number at the end of this process whose square is *exactly* 2, namely $\sqrt{2}$, having infinitely many decimals in its expansion. (Note that just because a number has infinitely many decimals, doesn't necessarily mean it isn't rational, think of $1/3 = 0.333\ldots$). We will explore the more technical details behind this in later chapters, but for now, we will simply say that the real numbers $\mathbb{R}$ are defined by the properties I – XVI, plus the property

XVII. *Any integer followed by an infinite sequence of decimals defines a valid real number,*

which is called the *completeness of the real numbers.*[1] We will not talk more of completeness in this chapter, since it is more of an analytic property of the real numbers rather than an algebraic one.[2]

From now on, we will assume nothing about the real numbers apart from the properties listed above in theorem 1.2. Even though you likely have previous knowledge of additional properties of real numbers, we approach the subject as if this is all we are allowed to use. Any other claims must be justified with proofs,

[1] See https://en.wikipedia.org/wiki/Completeness_of_the_real_numbers if you are interested in more details.

[2] When we say "analytic", we are referring to (real) analysis, which is the study of continuous things, which heavily relies on the completeness of the real numbers in particular. Algebra on the other hand doesn't care about the real numbers particularly, but more about solving equations and manipulating expressions, whether they contain integers, rationals or reals.

which we will give throughout the notes (unless the proofs are tedious/complicated and derail us from the topic at hand, often these will be proofs involving the notion of completeness).

In fact, to expand our toolbox, let us start by proving some easy results about real numbers which follow from theorem 1.2. All of these probably seem obvious to you, but how to prove them, allowing ourselves to only use I–XVI, is not always obvious!

Proposition 1.3. *Let $a, x, y \in \mathbb{R}$. Then*

i) *The numbers 0 and 1 are unique,*[3]

ii) *For each x, there is only one $-x$ and x^{-1} (where $x \neq 0$ for the latter),*

iii) $-(-x) = x$,

iv) $x + a = y + a \implies x = y$,

v) *If $a \neq 0$, then $xa = ya \implies x = y$,*

vi) $x \cdot 0 = 0 \cdot x = 0$,

vii) $x \cdot (-y) = (-x) \cdot y = -(x \cdot y)$,

viii) $(-x) \cdot (-y) = x \cdot y$,

ix) $(-1) \cdot x = -x$,

x) $(-1) \cdot (-1) = 1$.

Proof. For (i), suppose that there are two zeros, 0 and $\hat{0}$ both satisfying III, where $0 \neq \hat{0}$. Then by III,

$$0 = 0 + \hat{0} = \hat{0} + 0 = \hat{0},$$

contradicting that $0 \neq \hat{0}$. Replacing 0 and $\hat{0}$ with 1 and $\hat{1}$ above, the proof for 1 is the same by VIII.

For (ii), again suppose we have two different minus x's, $-x$ and $\ominus x$, both satisfying IV. Then by II, III and IV,

$$-x = -x + 0 = -x + (x + \ominus x) = (-x + x) + \ominus x = 0 + \ominus x = \ominus x,$$

contradicting that $-x \neq \ominus x$. A similar argument proves the uniqueness of x^{-1}.

[3] Meaning that 0 and 1 are the only numbers in $\mathbb{R}$ satisfying III and VIII of theorem 1.2 respectively.

For (iii), we have

$$\begin{aligned} -(-x) &= -(-x)+0 && \text{(by III)} \\ &= -(-x)+(-x+x) && \text{(by IV)} \\ &= (-(-x)+-x)+x && \text{(by II)} \\ &= 0+x && \text{(by IV)} \\ &= x && \text{(by III)} \end{aligned}$$

as required.

For (iv), suppose $x+a=y+a$. Then

$$\begin{aligned} x &= x+0 && \text{(by III)} \\ &= x+(a+-a) && \text{(by IV)} \\ &= (x+a)-a && \text{(by II)} \\ &+ (y+a)-a && \text{(by assumption)} \\ &= y+(a-a) && \text{(by II)} \\ &= y+0 && \text{(by IV)} \\ &= y && \text{(by III)} \end{aligned}$$

as required. A similar argument proves (v).

For (vi), we have

$$0+x\cdot 0 = x\cdot 0 = x\cdot(0+0) = (x\cdot 0)+(x\cdot 0)$$

by XI. Hence by (iv), $0 = x\cdot 0 = 0\cdot x$ by X.

For (vii), observe that

$$x\cdot y + x\cdot(-y) = x\cdot(y+-y) = x\cdot 0 = 0$$

by XI and (vi). Hence by (ii), $x\cdot(-y)$ is the unique inverse $-(x\cdot y)$ of $x\cdot y$. Similarly $(-x)\cdot y = -(x\cdot y)$.

Now for (viii), we have

$$(-x)\cdot(-y) = -((-x)\cdot y) = -(-(x\cdot y)) = x\cdot y$$

by (ii) and by (iii).

For (ix), observe that

$$x+(-1)\cdot x = x\cdot 1 + x\cdot(-1) = x\cdot(1+-1) = x\cdot 0 = 0$$

by VIII, XI and (vi). Thus by (ii), $(-1)\cdot x$ is the unique inverse $-x$ of x.

Finally (x) follows by (ix) with $x=-1$, and we get

$$(-1)\cdot(-1) = -(-1) = 1$$

by (iii). □

Notice that proving these "obvious" results is quite similar to playing chess in some sense; we know what tile (on the chessboard) we want to get to, but we can only make valid moves according to the rules of the game. In our case, the "valid moves" are the properties of theorem 1.2, and subsequent results we established from them.

Exercise 1.4. Prove the following for all $x, y \in \mathbb{R}$ where $x, y \neq 0$. You may use any of the facts from proposition 1.3.

i) $-0 = 0$,

ii) $1^{-1} = 1$,

iii) $(x^{-1})^{-1} = x$,

iv) $(x\cdot y)^{-1} = x^{-1}\cdot y^{-1}$.

Notation (Algebraic conventions)**.** We relax the rigid notation introduced in theorem 1.2 as follows.

- The product $x\cdot y$ is written simply as xy.
- The sum $x+-y$ is written simply as $x-y$.
- The product $x\cdot y^{-1}$ is written as $\frac{x}{y}$. We call x the *numerator* and y the *denominator*.
- $x+(y+z)=(x+y)+z$ is denoted simply as $x+y+z$, and similarly $x\cdot(y\cdot z) = (x\cdot y)\cdot z$ by xyz. Other similar relaxations are made when brackets are not necessary.

So for example, we write

$$\frac{3x-5y}{1+xyz} \quad \text{instead of} \quad (3\cdot x+-(5\cdot y))\cdot(1+x\cdot(y\cdot z))^{-1}.$$

Some consequences of our new notation are the familiar properties of proposition 1.5.

Proposition 1.5. *Let $w, x, y, z \in \mathbb{R}$. Then* *i)* $\frac{x}{y} \cdot \frac{w}{z} = \frac{xw}{yz}$, *ii)* $\frac{xy}{xz} = \frac{y}{z}$,

iii) $\frac{x}{z} + \frac{y}{z} = \frac{x+y}{z}$, *iv)* $\frac{x}{y} + \frac{w}{z} = \frac{xz+yw}{yz}$.

Proof. (i) follows from exercise 1.4(iv) since

$$\frac{x}{y} \cdot \frac{w}{z} = x \cdot y^{-1} \cdot w \cdot z^{-1} = (xw) \cdot (y^{-1} \cdot z^{-1}) = xw \cdot (yz)^{-1} = \frac{xw}{yz}.$$

(ii) then follows easily from (i): $\frac{xy}{xz} = \frac{x}{x} \cdot \frac{y}{z} = x \cdot x^{-1} \cdot \frac{y}{z} = 1 \cdot \frac{y}{z} = \frac{y}{z}$.

(iii) uses theorem 1.2(XI): $\frac{x}{z} + \frac{y}{z} = x \cdot z^{-1} + y \cdot z^{-1} = z^{-1} \cdot x + z^{-1} \cdot y = z^{-1} \cdot (x+y) = (x+y) \cdot z^{-1} = \frac{x+y}{z}$.

Finally (iv) follows by (ii) and (iii): $\frac{x}{y} + \frac{w}{z} = \frac{zx}{zy} + \frac{yw}{yz} = \frac{xz}{yz} + \frac{yw}{yz} = \frac{xz+yw}{yz}$. □

Notation (Integers and Rationals)**.** Since we are taking the properties of theorem 1.2 as our starting point, where the numbers 0 and 1 are defined by their behaviour, we had better discuss how other numbers like 2, 3, $^{22}/_{7}$ and -5 fit in to our framework.

For the natural numbers, we may take the natural definitions

$$\begin{aligned} 2 &:= 1+1 \\ 3 &:= 2+1 = 1+1+1 \\ 4 &:= 3+1 = 1+1+1+1 \\ &\vdots \end{aligned}$$

from these we can show things like "2 is unique", and using XV we get a sense of where each of them must lie on the number line (we can prove $0 \leqslant 1 \leqslant 2 \leqslant \cdots$).

The negative numbers are simply the additive inverses of the natural numbers in the sense of IV, so that takes care of them. Finally, as discussed in the previous notational remark, any rational $^{a}/_{b}$ is simply ab^{-1}, so that takes care of the rationals.

Exercise 1.6. Unless told otherwise, you may only use the facts of theorem 1.2 to justify your answers to the following.

1. Why is $x + x = 2x$?
2. Solve the equation $2x + 1 = 3$, justifying each step you make by referencing one of theorem 1.2, proposition 1.3 or 1.5.

3. Do the same for the equation $\frac{x}{2} + \frac{x}{3} = 3 + \frac{1}{3}$.
4. Prove that $(a+b)(c+d) = ac + bc + ad + bd$.
5. Prove that $\frac{-x}{y} = \frac{x}{-y} = -\frac{x}{y}$.
6. Show that $x \leqslant x$ for any $x \in \mathbb{R}$.
7. If $x \leqslant y$, we also write $y \geqslant x$. If $x \leqslant y$ and $x \neq y$, we write $x > y$ or $y < x$.

   What would the equivalents of XII–XVI be for $\geqslant$, $<$ and $>$?

   Prove that these equivalents are true using only theorem 1.2.
8. Rigorously prove that $\frac{6}{3} = 2$.
9. Show that if $x \leqslant y$ and $a \geqslant 0$, then $ax \leqslant ay$.
10. Assume that $0 < 1$.

    a) Show that if $x > 0$, then $x^{-1} > 0$.

    b) Show that $3 < \frac{22}{7} < 4$.

II. SQUARE ROOTS AND INDICES

IN this section, we study the quantity $x \cdot x$ and more generally, the repeated product $x \cdots x$ of a number x with itself. Just as with the last section, you most likely have previous knowledge of these ideas, but here we give a more rigorous treatment than you might be used to.

SQUARE ROOTS

In geometry, the area of a square is given by $y \cdot y$, where y is the length of a side. Consequently, we refer to the quantity $y \cdot y$ as y *squared*. The natural reversed question, "what is the side length of a square, given that its area is x?" gives rise to the idea of a square root.

Definition 2.1 (Square Root)**.** Let $x \in \mathbb{R}$. Any $y \in \mathbb{R}$ which satisfies the property

$$y \cdot y = x$$

is said to be a *square root* of x.

Example. 3 is a square root of 9, since $3 \cdot 3 = 9$. We also have that -3 is a square root of 9, since $(-3) \cdot (-3) = 9$.

Notice that we write "*a* square root", not "*the* square root" in definition 2.1, since using the definite article "*the*" implies that it is unique. In fact, as we have seen in the example, a square root is not unique: 9 has two square roots, 3 and -3.

Notation. For $y \in \mathbb{R}$, we abbreviate $y \cdot y$ to y^2.

Proposition 2.2. *If $x \in \mathbb{R}$, then $x^2 \geqslant 0$.*

Proof. By theorem 1.2(XIV), for any $x \in \mathbb{R}$, either $x \geqslant 0$ or $x \leqslant 0$.

If $x \geqslant 0$, then $x^2 = x \cdot x \geqslant 0$ by theorem 1.2(XVI).

If $x \leqslant 0$, then $x - x \leqslant 0 - x$, hence $0 \leqslant -x$. By XVI in exercise 1.6(7), we get $0 \leqslant (-x) \cdot (-x) = x^2$ by proposition 1.3(viii). □

Proposition 2.2 immediately gives us the following.

Corollary 2.3. *Let $x \in \mathbb{R}$. If $x < 0$, then x does not have a square root.*

Proof. By contradiction: suppose $x < 0$ and x does have a square root; call it y. Then by definition, $y^2 = x < 0$ (XVI in exercise 1.6(7)).

But also $y^2 \geqslant 0$ by proposition 2.2, contradicting that $y^2 < 0$. □

Exercise 2.4. Notice that we didn't prove any concrete inequalities in the previous section. In particular, we didn't even prove that $0 < 1$, which we assumed in question 10 of exercise 1.6. Prove it.

Thus no negative (< 0) real numbers have a square root. What about non-negative ($\geqslant 0$) real numbers? We will explore those in a moment, we first need this lemma.

Lemma 2.5 (Difference of two Squares)**.** *Let $x, y \in \mathbb{R}$. Then*

$$x^2 - y^2 = (x + y)(x - y).$$

Proof. Let $t = x + y$. Then

$$\begin{aligned}(x + y)(x - y) &= t(x - y)\\ &= xt - yt\\ &= x(x + y) - y(x + y)\\ &= x^2 + xy - yx - y^2\\ &= x^2 - y^2,\end{aligned}$$

as required. □

The following theorem really belongs in section 1.2 with all the other algebraic properties of the real numbers.

Theorem 2.6 ($\mathbb{R}$ is an Integral Domain)**.** *Let $x, y \in \mathbb{R}$ such that $xy = 0$. Then one of x, y must be zero.*

Proof. We prove this by contradiction. Suppose $x, y \in \mathbb{R}$ and $xy = 0$, but neither x nor y are zero. In particular since $x \neq 0$, the number x^{-1} exists. Thus

$$xy = 0 \implies x^{-1} \cdot xy = x^{-1} \cdot 0 \implies 1y = 0 \implies y = 0,$$

contradicting that $y \neq 0$. □

Examples 2.7. This important fact about real numbers allows us to solve equations such as $x(x - 1) = 0$, since it tells us that one of x or $(x - 1)$ must be zero, so we get either $x = 0$ or $x - 1 = 0 \implies x = 1$; and therefore the two solutions are $x = 0$ or $x = 1$.

Another example, consider the equation $x^2 - 16 = 0$. Since $16 = 4^2$, the left-hand side becomes $x^2 - 16 = x^2 - 4^2$, which by lemma 2.5 becomes $(x+4)(x-4)$. Thus the equation we have is $(x+4)(x-4)=0$, which by the theorem yields $x+4=0$ or $x-4=0$, i.e., $x=-4$ or $x=4$.

Now we address the question of square roots of non-negative real numbers.

Theorem 2.8. *Let $x \in \mathbb{R}$ such that $x \geqslant 0$. Then x has two square roots in $\mathbb{R}$, given by y and $-y$, where $y \in [0,\infty)$ is unique.*

Proof. We will not prove the existence of square roots in $\mathbb{R}$, because it is a consequence of the completeness property.[4] But the uniqueness part is easy. Indeed, suppose y and z are both square roots of x, but $y \neq z$. Then by definition, $y^2 = x$ and $z^2 = x$. In particular, $y^2 = z^2$, that is, $y^2 - z^2 = 0$, which by lemma 2.5 gives $(y+z)(y-z)=0$. But by theorem 2.6, this gives either $y+z=0$ or $y-z=0$, i.e., either $z=-y$ or $z=y$. Since we assumed that $z \neq y$, it follows that $z=-y$. This means that the only square root of x different from y is $-y$, as required. □

Notation. Let $x \geqslant 0$. Then we denote the unique non-negative y provided in theorem 2.8 by the symbol $\sqrt{x}$.

Example 2.9. As we saw earlier, 9 has two square roots, 3 and −3. The symbol $\sqrt{9}$ denotes the non-negative one, i.e., $\sqrt{9}=3$.

Notation is a very powerful tool in mathematics, and we will see throughout the course that many mathematical insights are purely a product of clever choice of notation. Simply giving a symbol to a concept allows us to express properties about it much more succinctly, and leads to theorems such as the following.

Theorem 2.10. *Let a, b, c be non-negative real numbers. Then*

i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$,

ii) $x\sqrt{a} \cdot y\sqrt{b} = xy\sqrt{ab}$ *for any* $x, y \in \mathbb{R}$,

iii) $\sqrt{abbc} = b\sqrt{ac}$.

Proof. For (i), observe that

$$(\sqrt{a}\sqrt{b}) \cdot (\sqrt{a}\sqrt{b}) = (\sqrt{a}\sqrt{a})(\sqrt{b}\sqrt{b}) = ab.$$

In particular, this means that $\sqrt{a}\sqrt{b}$ is a square root of ab since it agrees with definition 2.1, and moreover, $\sqrt{a}\sqrt{b}$ is non-negative since both $\sqrt{a}$ and $\sqrt{b}$ are

[4] This should be obvious in fact, since we already mentioned that $\mathbb{Q}$ satisfies all of I–XVI in theorem 1.2, but we know that $\sqrt{2}$ does not exist in $\mathbb{Q}$, thus the existence of $\sqrt{2}$ in $\mathbb{R}$ can only be a consequence of the additional property XVII (completeness).

(see XVI in theorem 1.2). But by theorem 2.8, the non-negative square root $\sqrt{ab}$ of ab is unique, so we must have $\sqrt{a}\sqrt{b} = \sqrt{ab}$.

(ii) and (iii) follow easily from (i):

$$x\sqrt{a} \cdot y\sqrt{b} = xy\sqrt{a}\sqrt{b} = xy\sqrt{ab},$$

$$\sqrt{abbc} = \sqrt{(bb)(ac)} = \sqrt{bb}\sqrt{ac} = b\sqrt{ac}$$

since $bb = bb \implies \sqrt{bb} = b$. □

Exercise 2.11. 1. Let $a, b \in \mathbb{R}$ with $a \geqslant 0$ and $b > 0$. Prove that

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

2. Prove that for all $x > 1/2$,

$$\sqrt{2x-1} + \frac{3}{\sqrt{2x-1}} = \frac{2(x+1)}{\sqrt{2x-1}}.$$

Why do we need the condition $x > 1/2$?

SURDS AND SURD FORM

At some point during your education—probably in primary school—you would have been told that it's "impolite" to leave, for example, $4/8$ as the answer to a maths problem, and you should instead write $1/2$ like a civilised member of society.

Here we introduce an analogue for square roots. Notice that, for example, $\sqrt{8}$ could be written as $\sqrt{2 \cdot 2 \cdot 2} = 2\sqrt{2}$. You are encouraged to "prefer" the latter. For larger examples, it might be clearer why this is preferable; e.g. $\sqrt{106\,722}$ is just $231\sqrt{2}$.

Notice that we are using theorem 2.10(iii) to simplify here, removing every pair of equal numbers below a square root and placing one outside. So for example, $\sqrt{72}$ is $6\sqrt{2}$ because $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$, so

$$\sqrt{72} = \sqrt{\underbrace{2 \cdot 2}\cdot 2 \cdot 3 \cdot 3} = \mathbf{2}\sqrt{2 \cdot \underbrace{3 \cdot 3}} = 2 \cdot \mathbf{3}\sqrt{2} = 6\sqrt{2}.$$

Alternatively, one could recognise that $72 = 4 \cdot 9 \cdot 2$, then by theorem 2.10(i), we get

$$\sqrt{72} = \sqrt{4 \cdot 9 \cdot 2} = \sqrt{4}\sqrt{9 \cdot 2} = 2\sqrt{9}\sqrt{2} = 2 \cdot 3\sqrt{2} = 6\sqrt{2}.$$

Either method is fine. But what makes $\sqrt{2}$ "unsimplifyable", where $\sqrt{72}$ wasn't? Is it because it is prime? Consider this example:

$$\sqrt{120} = \sqrt{4 \cdot 30} = 2\sqrt{30},$$

30 is not prime, but it cannot be reduced further; if we break it up into as many factors as we can (its *prime factorisation*), we get $30 = 2 \cdot 3 \cdot 5$. Thus by the reasoning before, we cannot "take out" any pairs from underneath the square root.

In fact, this is when an integer below a square root is "unsimplifyable": when its prime factorisation contains no repeated factors. We call these numbers *square-free*, and their square roots are called *surds*.

Definition 2.12 (Surd)**.** A *surd* is a positive real number of the form

$$\sqrt{p_1 \cdot p_2 \cdots p_n},$$

where p_i is prime for all $i = 1, \ldots, n$, and $p_i \neq p_j$ for $i \neq j$.

Examples 2.13. For example, $\sqrt{2}$ is a surd, since $\sqrt{2} = \sqrt{p_1}$ where $p_1 = 2$, and p_1 is prime. Similarly we have that $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \ldots$ are all surds.

Another example, $\sqrt{6}$ is a surd, since $\sqrt{6} = \sqrt{2 \cdot 3} = \sqrt{p_1 \cdot p_2}$, where $p_1 = 2$ and $p_2 = 3$, each p_i is prime, and $p_1 \neq p_2$, i.e., $p_i \neq p_j$ whenever i and j are different.

One final example, $\sqrt{1155}$ is a surd, since $\sqrt{1155} = \sqrt{3 \cdot 5 \cdot 7 \cdot 11}$, so $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, $p_4 = 11$, each p_i is prime, and $p_i \neq p_j$ when $i \neq j$.

$\sqrt{20}$ is *not* a surd, since $\sqrt{20} = \sqrt{2 \cdot 2 \cdot 5}$, so $p_1 = 2$, $p_2 = 2$, $p_3 = 5$. Each p_i is prime, but we have $p_1 = p_2$ even though $1 \neq 2$.

Theorem 2.14. *Every surd is irrational, that is, if x is a surd, then there are no two integers $a, b \in \mathbb{Z}$ such that $x = {}^{a}/_{b}$ where $b \neq 0$.*

Proof. A proof similar to the irrationality of $\sqrt{2}$ in appendix A can be adapted to prove this more general theorem. □

Definition 2.15 (Linear Combination over $\mathbb{Q}$)**.** Let $\{x_1, x_2, \ldots, x_n\} \subseteq \mathbb{R}$. Then a *linear combination over* $\mathbb{Q}$ *of* $\{x_1, x_2, \ldots, x_n\}$ is a real number of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

where $a_1, a_2, \ldots, a_n \in \mathbb{Q}$.

Example 2.16. If $S = \{\pi, \sqrt{2}, 1-\sqrt{5}\}$, then some rational linear combinations of these numbers are

$$2\pi + 3\sqrt{2} + 7(1-\sqrt{5})$$
$$3\pi - \frac{\sqrt{2}}{2} - (1-\sqrt{5}) = 3\pi + (-\tfrac{1}{2})\sqrt{2} + (-1)(1-\sqrt{5})$$
$$\pi + \sqrt{2} = 1\pi + 1\sqrt{2} + 0(1-\sqrt{5})$$

Definition 2.17 (Surd Form)**.** Let $\mathbb{S}$ denote the set of all surds. A real number is said to be in *surd form* if it is expressed as a linear combination over $\mathbb{Q}$ of $\mathbb{S} \cup \{1\}$.

Examples 2.18. We give some examples of how we may transform some real numbers into surd form.

i) $\frac{1}{2} + 5\sqrt{2}$ is in surd form, since it equals $\frac{1}{2} \cdot 1 + 5 \cdot \sqrt{2}$, and $\frac{1}{2}, 5 \in \mathbb{Q}$ and $1, \sqrt{2} \in \mathbb{S} \cup \{1\}$.

ii) $\sqrt{60} + \sqrt{800}$ is not in surd form, since $\sqrt{60}, \sqrt{800} \notin \mathbb{S}$, because $60 = 2^2 \cdot 3 \cdot 5$ and $800 = 2^5 \cdot 5^2$, and therefore $\sqrt{60}$ and $\sqrt{800}$ is not made up of a product of *unequal* primes ($p_i \neq p_j$) under the square root. However using theorem 2.10(iii), we have

$$\sqrt{60} = \sqrt{2 \cdot 2 \cdot 3 \cdot 5} = 2\sqrt{3 \cdot 5} = 2\sqrt{15},$$

where $\sqrt{15} \in \mathbb{S}$, and similarly

$$\begin{aligned}\sqrt{800} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5} &= 2\sqrt{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5} \\ &= 2 \cdot 2\sqrt{2 \cdot 5 \cdot 5} \\ &= 2 \cdot 2 \cdot 5\sqrt{2} = 20\sqrt{2}.\end{aligned}$$

Thus we may write $\sqrt{60} + \sqrt{800}$ in surd form as $2\sqrt{15} + 20\sqrt{2}$, since $2, 20 \in \mathbb{Q}$ and $\sqrt{15}, \sqrt{2} \in \mathbb{S} \cup \{1\}$.

iii) $\frac{7-5\sqrt{3}}{3}$ is nearly in surd form, however to write it more precisely as a linear combination over $\mathbb{Q}$, we express it as $\left(\frac{7}{3}\right)1 + \left(-\frac{5}{3}\right)\sqrt{3}$, this way it is clear that $\frac{7}{3}, -\frac{5}{3} \in \mathbb{Q}$, and $1, \sqrt{3} \in \mathbb{S} \cup \{1\}$.

iv) $(\sqrt{2} + 5\sqrt{3})(7\sqrt{12} - \sqrt{18})$ is not in surd form. We can start by observing that the second term has square roots which can be reduced.

$$7\sqrt{12} - \sqrt{18} = 7\sqrt{2 \cdot 2 \cdot 3} - \sqrt{2 \cdot 3 \cdot 3} = 14\sqrt{3} - 3\sqrt{2},$$

Hence we have the product $(\sqrt{2}+5\sqrt{3})(14\sqrt{3}-3\sqrt{2})$. But this is still not in surd form: it's a product, not a linear combination of surds. Exercise 1.6(4) and theorem 2.10(ii) can help us in expanding this out:

$$\begin{aligned}&(\sqrt{2}+5\sqrt{3})(14\sqrt{3}-3\sqrt{2})\\&=14\sqrt{2\cdot 3}-3\sqrt{2\cdot 2}+5\sqrt{3\cdot 3}-5\cdot 3\sqrt{3\cdot 2}\\&=14\sqrt{6}-3\cdot 2+5\cdot 3-15\sqrt{6}\\&=204-\sqrt{6},\end{aligned}$$

which is in surd form.

v) $\frac{5}{\sqrt{5}}+\frac{\sqrt{5}}{5}$ is not in surd form, the first term $\frac{5}{\sqrt{5}}$ is neither a surd nor rational. However multiplying the numerator and denominator by $\sqrt{5}$, we get $\frac{5\sqrt{5}}{\sqrt{5}\sqrt{5}}$, which by definition of $\sqrt{\ }$ becomes $\frac{5\sqrt{5}}{5}$. Therefore we have $\frac{5\sqrt{5}}{5}+\frac{\sqrt{5}}{5}$, which is equal to $\frac{6\sqrt{5}}{5}$, or $\frac{6}{5}\sqrt{5}$, which is in surd form.

When surds appear in the denominator, it is not immediately clear whether that number can be expressed in surd form (unlike a surd in the numerator; $\frac{\sqrt{5}}{5}$ is clearly just $\frac{1}{5}\sqrt{5}$). Thus surds should always be "removed" from the denominator. This process of removing surds from the denominator is called *rationalising the denominator* (since the denominator becomes rational as a consequence).

vi) $\frac{2}{\sqrt{2}+\sqrt{3}}$ is not in surd form. If we try the same technique as we did in example (iv), that is, multiplying the numerator and denominator by what is in the denominator, it will not work, since multiplying the denominator by itself does not entirely get rid of square roots: $(\sqrt{2}+\sqrt{3})(\sqrt{2}+\sqrt{3})=2+3+2\sqrt{6}$. This is due to the fact that $(a+b)(a+b)=a^2+b^2+\boldsymbol{2ab}$. But if we instead consider what lemma 2.5 gives us, i.e., $(a+b)(a-b)=a^2-b^2$, notice that no terms appear here without being squared. So multiplying the numerator and denominator by the denominator with *one of the signs reversed*, we get

$$\frac{2}{\sqrt{2}+\sqrt{3}}=\frac{2(\sqrt{2}-\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}=\frac{2\sqrt{2}-2\sqrt{3}}{-1}=2\sqrt{3}-2\sqrt{2},$$

which is now clearly in surd form.

vii) $5\sqrt{2/3}$ is not in surd form. But by exercise 2.11, we can express

$$5\sqrt{\frac{2}{3}}=5\frac{\sqrt{2}}{\sqrt{3}},$$

and so $5\sqrt{3/2} = 5\frac{\sqrt{2}\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{5\sqrt{6}}{3}$, which is in surd form.

Remark 2.19. Throughout the rest of the course, you are encouraged to give answers to problems in surd form whenever possible. Most importantly, you should simplify any square roots of numbers which are not square-free.

Most modern calculators can do this automatically for you, but only for integers that are sufficiently small. Try to simplify $\sqrt{160\,083}$ using your calculator, it will probably give you something like 400.1037, instead of the exact form $321\sqrt{3}$, which you should be able to obtain by hand.

Exercise 2.20. 1. Express the following in surd form.

a) $\sqrt{243}$

b) $8\sqrt{1250}$

c) $\sqrt{44}$

d) $\dfrac{1+\sqrt{2}}{\sqrt{3}}$

e) $\sqrt{60}$

f) $\sqrt{500}-\sqrt{124}+5\sqrt{49}$

g) $\dfrac{6+\sqrt{2}}{\sqrt{2}+7}$

h) $(2\sqrt{12}-3)\dfrac{3}{4-\sqrt{2}}$

i) $(4\sqrt{7}+3)(4\sqrt{7}+3)$

j) $(4\sqrt{7}+3)(4\sqrt{7}-3)$

k) $\dfrac{\sqrt{2}}{\sqrt{2}-1}$

l) $\dfrac{4}{1+\sqrt{2}}$

m) $24\sqrt{400}$

n) $\dfrac{2\sqrt{2}+3\sqrt{3}+5\sqrt{5}}{6\sqrt{6}+7\sqrt{7}}$

o) $5+\sqrt{18}+9-\sqrt{36}$

p) $(2\sqrt{2}-3\sqrt{3})(\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}})$

2. Simplify the expression $\dfrac{1}{\sqrt{3}}-\dfrac{\sqrt{3}}{1-\sqrt{3}}$ writing your answer in the form $(a+b\sqrt{3})/c$, where $a,b,c\in\mathbb{Z}$. (MATSEC Sept '15)

3. Express $\dfrac{\sqrt{2}-1}{2\sqrt{2}+3}+\dfrac{3}{2\sqrt{2}-3}$ in the form $(a+b\sqrt{2})$. (MATSEC May '15)

4. Notice that in examples 2.18(iv), we simplified the second term in the product so that they were both in surd form, and subsequently multiplied them. Once we multiplied and combined like-terms, all

square roots which appeared were surds—we didn't need to simplify further. Is it always the case that if we start with two numbers in surd form, their product (expanded and like-terms combined) will be in surd form?

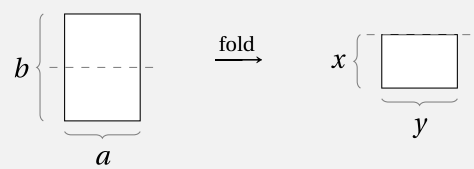
5. Express $\dfrac{1}{\sqrt{2}+2\sqrt{3}+3\sqrt{5}}$ in surd form.

6. Solve the equation

$$\frac{1}{\sqrt{2}+\sqrt{3}} = \frac{1}{x} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}},$$

giving your answer in surd form.

7. A-series paper (A4, A3, ...) has the property that folding it in half preserves the ratio of the sides.



In other words, we have $a:b$ as $x:y$ in the diagram above. Show that this can only happen if the ratio is $1:\sqrt{2}$.

8. Prove that

$$\frac{(1+\sqrt{2})(3-\sqrt{2})(5+\sqrt{2})}{(7-\sqrt{2})(9+8\sqrt{2})(13-2\sqrt{2})} = \frac{1}{p},$$

where p is prime.

9. The area of a triangle is $13\,\text{cm}^2$, and its base is $4-\sqrt{3}$ cm wide. What is the triangle's height?

10. Solve the following equation for x.

$$a = \frac{\sqrt{2}+\frac{\sqrt{3}}{x+1}}{\sqrt{2}-\frac{\sqrt{3}}{x+1}}$$

Hence or otherwise, determine the value(s) of $a \in \mathbb{R}$ for which it has no solutions.

INDICES

Just as n copies of x added together $(x + \cdots + x)$ may be written as nx (recall exercise 1.6(1)), here we introduce a shorthand for multiplication.

Definition 2.21 (Power[5])**.** Let $n \in \mathbb{N}$. For any $x \in \mathbb{R}$, we define the notation x^n by

$$x^n := \underbrace{x \cdot x \cdots x}_{n \text{ times}},$$

where n here is called a *power* (or *index* or *exponent*) of x.

This definition immediately gives us the following theorem.

Theorem 2.22 (Laws of Indices)**.** *Let $a, b \in \mathbb{N}$, and let $x, y \in \mathbb{R}$. We have the following laws.*

I. $x^a \cdot x^b = x^{a+b}$

II. $\dfrac{x^a}{x^b} = x^{a-b}$, *for $a > b$*

III. $(x^a)^b = x^{ab}$

IV. $(xy)^a = x^a y^a$, *for x and y not both negative*

V. $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$

Proof. We give the proof of law III here, the other four can easily be proved in a similar fashion.

$$\begin{aligned}
(x^a)^b &= \underbrace{x^a \cdot x^a \cdots x^a}_{b \text{ times}} && \text{(by definition 2.21)} \\
&= \underbrace{\underbrace{x \cdot x \cdots x}_{a \text{ times}} \cdot \underbrace{x \cdot x \cdots x}_{a \text{ times}} \cdots \underbrace{x \cdot x \cdots x}_{a \text{ times}}}_{b \text{ times}} && \text{(again by definition 2.21)} \\
&= \underbrace{x \cdot x \cdots x}_{ab \text{ times}} = x^{ab}, && \text{(again by definition 2.21)}
\end{aligned}$$

as required. □

These "laws" are mere immediate consequences of the definition: for example, the first one says that if you write down two x's, and then three x's, you get five x's:

$$x^2 \cdot x^3 = xx \cdot xxx = xxxxx = x^5,$$

[5] The definition here is not entirely precise, a more formal definition would be expressed *recursively*. This is something we will revisit later in section 6.2. In general, when one sees dots $(\cdots)$ in a mathematical definition/proof, it's a sign that things are a bit hand-wavy!

very straightforward stuff.

Observe that the power notation we have introduced is consistent with the notation of squaring $y \cdot y$ which we denoted by y^2 in section 2.1.

Now, as mathematicians, we would like to generalise our notation. Let us start by allowing for integer powers, particularly zero and negative whole numbers. In principle, we can define our notation anyway we like, but it would be nice if our new definition was such that I–V in theorem 2.22 remain true.

If we want I to remain true, then for any x, we must have $x^1 \cdot x^0 = x^{1+0} = x^1$, i.e., $x \cdot x^0 = x$, i.e., $x(x^0 - 1) = 0$. Now we invoke theorem 2.6. If x is zero, then this equation is obviously true, but for non-zero x, we must have $x^0 = 1$. Thus we will set $x^0 = 1$ for all $x \in \mathbb{R}$, even in the zero case. (Some people like to leave 0^0 undefined, but in the rare cases where we might have to deal with it, rather than awkwardly avoiding it, it is often convenient to just have it equal to 1.)

By the same reasoning, we would want that $x^n \cdot x^{-n} = x^{n-n} = x^0 = 1$, which would imply that $x^{-n} = 1/x^n$, and so we take this as our definition for negative powers. Notice this also implies that 0^{-n} is not defined for $n \neq 0$, since $0^n = 0$ for all $n \in \mathbb{N}$. In summary, we have the following definition.

Definition 2.23 (Integer Power)**.** Let $x \in \mathbb{R}$, and $n \in \mathbb{Z}$. We extend the definition of x^n, by defining

$$x^n := \begin{cases} \underbrace{x \cdot x \cdot x \cdots x}_{n \text{ times}} & \text{if } n > 0 \\ 1 & \text{if } n = 0 \\ \dfrac{1}{\underbrace{x \cdot x \cdot x \cdots x}_{-n \text{ times}}} & \text{if } n < 0 \text{ and } x \neq 0. \end{cases}$$

With this definition, the results of theorem 2.22 are true for any $a, b \in \mathbb{Z}$. For example,

$$3^8 \cdot 3^{-5} = 3^8 \cdot \frac{1}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = \frac{\cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot 3 \cdot 3 \cdot 3}{\cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3}} = 3 \cdot 3 \cdot 3 = 3^{8+(-5)},$$

showing that law I holds.

Next, we want to generalise our notation even further, allowing for rational powers. Again, in principle, we can define a more general notation any way we'd like, but it would be nice if the laws in theorem 2.22 remain true. Take for instance, $3^{4/5}$. What should this equal? Well if III is to hold, we would have that

$$(3^{4/5})^5 = 3^{\frac{4}{5}\cdot 5} = 3^4.$$

In other words, $3^{4/5}$ should solve $x^5 = 3^4$. Now we borrow a fact from real analysis, which is a generalisation of theorem 2.8. We will not prove it here, since as usual, it is a consequence of completeness of $\mathbb{R}$.

Fact 2.24. *Let $n \in \mathbb{N}$ and $A \in \mathbb{R}$. Then*

(i) if n is odd, then there is a unique number $x \in \mathbb{R}$ such that $x^n = A$.

(ii) if n is even and $A \geqslant 0$, then there is a unique real number $x > 0$ such that both $x^n = A$ and $(-x)^n = A$.

(iii) if n is even and $A < 0$, then there is no real number x satisfying $x^n = A$.

It follows from (i) of the fact that $3^{4/5}$ must be the unique real number satisfying $x^5 = 3^4$. This number is called the fifth root of 3^4, which we denote by $\sqrt[5]{3^4}$. More generally, if n is odd, then we define

$$x^{a/n} := \sqrt[n]{x^a},$$

where $\sqrt[n]{A}$ denotes the unique solution to $x^n = A$.

Now if n is even, things are not as straightforward. Indeed, consider $3^{1/2}$. What should this equal? Well by the same reasoning as before, we conclude that it should solve $x^2 = 3$. But by the fact, we still have two choices, namely x and $-x$ which (ii) provides. To relieve ambiguity, we usually pick the positive one, and denote it by $\sqrt{3}$ (as in definition 2.1). This is called the *principal square root* of 3. In general, if $A \geqslant 0$, then $\sqrt[n]{A}$ denotes the positive solution to $x^n = A$, which we call the *principal nth root of A*, and if $A < 0$, then the notation is undefined. Therefore for $a/n \in \mathbb{Q}$ where $a \in \mathbb{Z}$ and $n \in \mathbb{N}$, we define

$$x^{a/n} := \sqrt[n]{x^a}.$$

Our decision to choose the positive root sometimes causes problems. For instance,

$$-5 = (-5)^1 = (-5)^{2\cdot\frac{1}{2}} \stackrel{?}{=} ((-5)^2)^{1/2} = 25^{1/2} = 5,$$

which is clearly wrong. Indeed, we can trace the problem to the equality $\stackrel{?}{=}$, which comes from III of theorem 2.22. So we need to be quite careful, it seems that choosing the positive root to be the value we take for even denominators has invalided III for negative bases x. (Notice that were $25^{1/2}$ to equal -5, there would be no problem above). This is analogous to the fact that $\sqrt{x^2}$ is not always equal to x.

The real picture is the following: in a lot of situations, it is perfectly fine to adopt the convention that $x^{a/n}$ denotes the principal root $\sqrt[n]{x^a}$. Usually in such scenarios, we do not really care *which* of the two roots we have, we simply want a number y which satisfies $y^n = x^a$. Other times however, we might get paradoxes which arise when the two are interchanged, which is what occurred above.

Thus, for rational powers, the laws in theorem 2.22 can be shown to be true, with some caveats which crop up in specific instances, which are consequences of our choice of convention.

Later on, we see that the number x^y can be defined for all powers $y \in \mathbb{R}$, and even for y in sets larger than $\mathbb{R}$; always in such a way that all the definitions here are respected, and that the results of theorem 2.22 still hold (adjusting for minor caveats as we encountered above). For now, we will simply give some intuition as to what this "should" be when $x > 0$.

We will see later that we want things of the form x^y to be continuous in y. All this means, intuitively, is that if we vary y by a small quantity, then x^y will also change by a reasonably small quantity. Thus,

$$3^{\sqrt{2}},$$

for example, should be reasonably close to $3^{1.4}$ (which we can evaluate since 1.4 is rational) closer to $3^{1.414}$, even closer to $3^{1.41412135}$, and so on, so that the more digits we take in the power, the more accurate our approximation. The number $3^{\sqrt{2}}$ should be thought of as the "completion" of this process, just as $\sqrt{2}$ itself is the completion of a similar process.

Exercise 2.25. 1. Evaluate the following WITHOUT A CALCULATOR.

a) 2^3	b) 4^{-2}	c) $[2.314\times 10^{-37}]^0$
d) $5^{2/3}5^{4/3}$	e) 2^{-9}	f) $64^{-\frac{1}{4}}$
g) $16^0 \times 243^{1/5}$	h) $\pi^2\pi^{-3}\times\pi 5^2$	i) 153^2-47^2
j) 2^{2^3}	k) $(\frac{3}{7})^2(1/49)^{-1/2}$	l) $22-4^3(\frac{1}{2})^3$
m) $(-1)^{207841}(4)^{1/2}$	n) $(\frac{1}{2})^{3/2}$	o) $\sqrt{\sqrt{\sqrt{900}}}$

2. a) Prove that $\sqrt[3]{ab}=\sqrt[3]{a}\sqrt[3]{b}$ in the style of theorem 2.10.

   b) Similarly prove that $\sqrt[3]{abbbc}=b\sqrt[3]{ac}$.

   c) Allowing yourself to use ideas from this section, how could you prove these two results for any nth root easily? (You may assume that $a,b,c\geqslant 0$ in the case that n is even.)

3. Solve the equation $x^2=32x^{-2}$.

4. How would you generalise the definition of surds and surd form for cube roots ($\sqrt[3]{\ }$)? Express

$$\frac{1}{\sqrt[3]{2}+\sqrt[3]{3}}$$

is your new surd form.

[Hint: look up the “sum of two cubes” formula in the A-level booklet!]

III. QUADRATICS

HOW can we solve equations involving an unknown x, together with different powers of x, such as x^2, x^3, and so on? This turns out to be a difficult problem in general, let us start first with the case where we allow x and x^2 to appear in an equation. We call these quadratic equations.

SOLVING QUADRATIC EQUATIONS

Definition 3.1 (Quadratic)**.** A *quadratic* is an algebraic expression of the form

$$ax^2 + bx + c,$$

where x is a variable, and $a, b, c \in \mathbb{R}$ and $a \neq 0$. An equation $\phi(x) = 0$ where ϕ is a quadratic is said to be a *quadratic equation* (QE).

Definition 3.2 (Root)**.** Let $F \subseteq \mathbb{R}$ and let $\phi(x)$ be an algebraic expression dependent on x. A *root* or *zero* of ϕ is an element $x \in F$ such that $\phi(x) = 0$.

In particular, we say that x is a *real root* if we put $F = \mathbb{R}$.

Sometimes if we have a quadratic equation $\phi(x) = 0$, we call its solutions its *roots* or its *zeros*, in view of the definition above.

Examples 3.3. $x^2 - 1$, $5x^2 - x$, and $3x^2 + 5x - 12$ are three examples of *quadratics*, whereas $x^2 - 1 = 0$, $5x^2 - x = 0$, and $3x^2 + 5x - 12 = 0$ are examples of *quadratic equations*.

$x = 1$ is a *root* of $x^2 - 1$ since $(1)^2 - 1 = 1 - 1 = 0$, it is also a *solution* of $x^2 - 1 = 0$.

Notation. To abbreviate "$x = a$ or $x = -a$", we write $x = \pm a$.

Remark 3.4. Sometimes quadratic equations can be solved if they are transformed into products of linear factors, as we did in examples 2.7. This is called *factorisation*, and can be done by close inspection of the quadratic. A few examples:

$$x^2 - 9 = 0 \implies (x+3)(x-3) = 0 \implies x = \pm 3$$
$$x^2 - 5x + 6 = 0 \implies (x-2)(x-3) = 0 \implies x = 2,\ x = 3$$
$$6x^2 + 7x - 3 = 0 \implies (3x-1)(2x+3) = 0 \implies x = \tfrac{1}{3},\ x = -\tfrac{3}{2}$$

This cannot always be done, however. E.g., try to factorise $x^2 + x + 1$. (We will see a criterion for when this possible in the next section, theorem 3.16.)

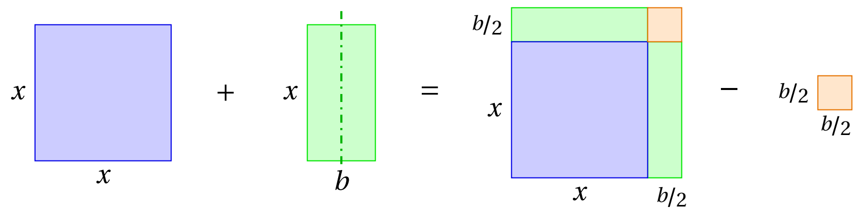


FIGURE 1: Visualisation of Completing the Square

Lemma 3.5. *Suppose $x, a \in \mathbb{R}$ and $a \geqslant 0$. Then*

$$x^2 = a \iff x = \pm\sqrt{a}.$$

Proof. This follows immediately from theorem 2.8. □

The following result is a fundamental identity which you are encouraged to commit to memory!

Theorem 3.6 (Completing the square)**.** *Let $b \in \mathbb{R}$. Then for all $x \in \mathbb{R}$,*

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2.$$

Proof. Expanding the right-hand side,

$$\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = x^2 + bx.$$ □

Exercise 3.7. Refer to the illustration in figure 1. Provide a geometric argument for theorem 3.6 (you may assume $b > 0$).

Examples 3.8. This theorem allows us to solve *any* quadratic equation!

(i) Suppose we have $x^2 - 4x + 1 = 0$. Applying the theorem to the left-hand side we get

$$x^2 - 4x + 1 = (x-2)^2 - (-2)^2 + 1 = (x-2)^2 - 3.$$

Thus $x^2 - 4x + 1 = 0$ becomes $(x-2)^2 - 3 = 0$, i.e., $(x-2)^2 = 3$, which by lemma 3.5 becomes $x - 2 = \pm\sqrt{3}$, so the solutions are $x = 2 \pm \sqrt{3}$.

(ii) Let us give a more complicated example. We solve $7x^2 - 5x - 6 = 0$. Since the coefficient of x^2 is not 1, we cannot immediately complete the square, since the identity calls for something of the form $x^2 + bx$. But since this is an equation, we can divide throughout by 7 to get

$$\begin{aligned} & x^2 - \tfrac{5}{7}x - \tfrac{6}{7} = 0 \\ \implies\quad & \left(x - \tfrac{5}{14}\right)^2 - \tfrac{25}{14^2} - \tfrac{6}{7} = 0 && \text{(completing the square)} \\ \implies\quad & \left(x - \tfrac{5}{14}\right)^2 - \tfrac{193}{14^2} = 0 && \text{(LCM)} \\ \implies\quad & \left(x - \tfrac{5}{14}\right)^2 = \tfrac{193}{14^2} \\ \implies\quad & x - \tfrac{5}{14} = \pm\tfrac{\sqrt{193}}{14} && \text{(lemma 3.5)} \\ \implies\quad & x = \tfrac{5\pm\sqrt{193}}{14}, \end{aligned}$$

which are the two solutions.

(iii) Completing the square is not just useful for equation solving, as we shall see later. In general, it is quite useful to be able to change a quadratic $ax^2 + bx + c$ to something of the form $a(x + p)^2 + q$, and this is precisely what completing the square allows us to do.

Say we want to rewrite $10x^2 - 2x + 1$ in this form. Since we don't have an equation as we did in (ii), we can't divide by 10. We can still factorise 10 out though:

$$\begin{aligned} 10x^2 - 2x + 1 &= 10\left(x^2 - \tfrac{1}{5} + \tfrac{1}{10}\right) \\ &= 10\left(\left(x - \tfrac{1}{10}\right)^2 - \tfrac{1}{100} + \tfrac{1}{10}\right) \qquad \text{(completing the square)} \\ &= 10\left(\left(x - \tfrac{1}{10}\right)^2 + \tfrac{9}{100}\right) \\ &= 10\left(x - \tfrac{1}{10}\right)^2 + \tfrac{9}{10}, \end{aligned}$$

as required.

(iv) We can use completing the square to prove things like the following: that $2x^2 - x + 1$ is always a positive quantity. In it's current form, it's not clear that the quantity is always positive. But by completing the square,

$$\begin{aligned} 2x^2 - x + 1 = 2\left(x^2 - \tfrac{1}{2}x + \tfrac{1}{2}\right) = 2\left(\left(x - \tfrac{1}{4}\right)^2 + \tfrac{7}{16}\right) &= 2\underbrace{\left(x - \tfrac{1}{4}\right)^2}_{\geqslant 0} + \tfrac{7}{8} \\ &\geqslant 2 \cdot 0 + \tfrac{7}{8} = \tfrac{7}{8}. \end{aligned}$$

Indeed, since the quantity $x - \frac{1}{4}$ is being squared, the least it can be is zero (by proposition 2.2), and moreover, we are adding $\frac{7}{8}$, so the given quadratic expression is at least $\frac{7}{8}$ for any value of x.

Examples 3.9. In these examples, we solve a few different quadratic equations in which minor complications arise. To keep things simple, all the quadratics given here are solvable by factorising—but the reasoning applies to any QEs, and equation solving problems in general.

i) $12x^2 + 2x - 4 = 0$.

The first thing we notice is that there is a common factor of 2, so dividing both sides of the equation by 2 (i.e., multiplying throughout by $1/2$) will give an equation whose left-hand side might prove easier to factorise.

$$\begin{aligned} & 6x^2 + x - 2 = 0 \\ \implies & (3x+2)(2x-1) = 0 \\ \implies & x = -2/3, x = 1/2 \end{aligned}$$

ii) $17x^2 + 81x - 20 = 0$.

The coefficient 17 of x^2 might make this seem harder at first, but being prime, this actually gives us fewer options for factorisation. The left hand side factorises to $(17x-4)(x+5) = 0$, and so the solutions are $x = 4/17$, $x = -5$.

iii) $28x^2 + 7x - 7 = 7x^2 + 5x + 1$.

Horror! We don't have $= 0$ on the end! This is quite simple to get around as I'm sure you guessed, we can move everything over to the other side and solve normally (this actually corresponds to adding $-7x^2$, $-5x$ and -1 to both sides, if we reason about it in the section 1.2 sense).

In fact, so we don't forget about section 1.2, let's justify each step in solving this equation properly:

$$\begin{aligned} & 28x^2 + 7x - 7 = 7x^2 + 5x + 1 \\ \implies & (28x^2 + 7x - 7) - 7x^2 = (7x^2 + 5x + 1) - 7x^2 \\ \implies & 28x^2 - 7x^2 + 7x - 7 = 7x^2 - 7x^2 + 5x + 1 \\ \implies & (28-7)x^2 + 7x - 7 = 0 + 5x + 1 \\ \implies & 21x^2 + 7x - 7 = 5x + 1 \\ & \vdots \\ \implies & 21x^2 + 2x - 8 = 0 \\ \implies & (7x-4)(3x+2) = 0 \\ \implies & 7x - 4 = 0 \quad \text{or} \quad 3x + 2 = 0 \end{aligned}$$

$$\begin{aligned}
\implies && 7x = 4 \quad &\text{or} \quad 3x = -2 \\
\implies && \tfrac{1}{7}7x = \tfrac{1}{7}4 \quad &\text{or} \quad \tfrac{1}{3}3x = \tfrac{1}{3}(-2) \\
\implies && x = \tfrac{4}{7} \quad &\text{or} \quad x = -\tfrac{2}{3},
\end{aligned}$$

as you can see, we did skip some step justifications here. Perhaps this will cause you to appreciate how many algebraic steps we are used to applying automatically—we usually solve something like this in 3 to 4 steps!

iv) $x^2 - 4 = 0$.

This can be solved in three ways: we could either recognise the left-hand side as a difference of two squares, and do

$$(x-2)(x+2) = 0 \implies x = 2, x = -2.$$

Alternatively, we could simply take the 4 over to the right hand side and by lemma 3.5 we get

$$x^2 = 4 \implies x = \pm\sqrt{4} = \pm 2,$$

or we could look at it as we do the usual quadratic with a non-zero coefficient of x:

$$x^2 + 0x - 4 = 0 \implies (x-2)(x+2) = 0 \implies x = 2, x = -2.$$

v) $x^2 - 3x = 0$.

This is quite an easy equation to solve, one simply has to notice that x is a common factor of the left-hand side, and do

$$x^2 - 3x = 0 \implies x(x-3) = 0 \implies x = 0, x = 3.$$

Perhaps a word of caution: notice that in the very first example, we divided by 2 because it was a common factor of the left-hand-side. Why not divide by x here then?

Well, perhaps the way we should look at "dividing by 2" is the following:

$$\begin{aligned}
12x^2 + 2x - 4 = 0 &\implies 2(6x^2 + x - 2) = 0 \\
&\implies 2 = 0 \quad \text{or} \quad 6x^2 + x - 2 = 0.
\end{aligned}$$

Clearly $2 = 0$ is a nonsensical conclusion, so we dismiss it and focus on the other solution. (After all, what we conclude when we apply theorem 2.6 is an OR statement, meaning at least one of the two conclusions is true, not necessarily both).

And in fact, it is precisely the fact that 2 is not 0 which allows us to divide by 2 in the first place (which is actually *multiplication by* 2^{-1}). In the case where we have $x(x-3)=0$, we cannot divide by x (i.e., multiply by x^{-1}) if x could possibly be zero: x^{-1} does not even exist in that case! (by theorem 1.2). Thus we should always think of "dividing both sides of an equation" in the sense of the above, where we get silly conclusions such as $2=0$ which we discard. This way, we never lose any possible solutions which we would miss by "dividing".

vi) $x^5-13x^3+36x=0$.

"This isn't a quadratic!", you might protest. But let's give it a go. First of all, notice that just as in the previous example, x is a common factor:

$$x(x^4-13x^2+36)=0 \implies x=0 \quad \text{or} \quad x^4-13x^2+36=0,$$

so we already found a solution. Let's focus on $x^4-13x^2+36=0$ now. What could this give us, and more importantly, how do we solve it? If we stare at it long enough, we notice that even though it's not exactly a quadratic, it has a quadratic form. In fact, if we let $t=x^2$, then the equation is simply $t^2-13t+36=0$.

Thus we have

$$\begin{aligned}
&t^2-13t+36=0\\
\implies &(t-4)(t-9)=0\\
\implies &t=4 \quad \text{or} \quad t=9\\
\implies &x^2=4 \quad \text{or} \quad x^2=9\\
\implies &x=\pm 2 \quad \text{or} \quad x=\pm 3
\end{aligned}$$

by lemma 3.5. Thus the solutions are $x=0, x=\pm 2, x=\pm 3$. The introduction of a new variable t makes things clear (just as in, say, the proof of lemma 2.5), but it is unnecessary. We could work without it:

$$\begin{aligned}
&x^4-13x^2+36=0\\
\implies &(x^2)^2-13x^2+36=0\\
\implies &(x^2-4)(x^2-9)=0\\
\implies &x^2=4 \quad \text{or} \quad x^2=9\\
\implies &x=\pm 2 \quad \text{or} \quad x=\pm 3.
\end{aligned}$$

Notice we get 5 solutions to this equation.

vii) $2x = \sqrt{7-27x}$.

Again, this isn't clearly a quadratic: but remember, what is the definition of the $\surd$ symbol? Well, it means that $2x$ is a square root of $7-27x$, so by definition 2.1,

$$\begin{aligned} &(2x)^2 = 7-27x \\ \implies &4x^2 + 27x - 7 = 0 \\ \implies &(4x-1)(x+7) = 0 \\ \implies &x = \tfrac{1}{4} \quad \text{or} \quad x = -7. \end{aligned}$$

Even though we did this with none of the other examples, let's check that these answers are right. What does it mean that these are "answers"? Well, it means that substitution in the left-hand side and the right-hand side of the given equation should make them equal. Let's start with $x = \frac{1}{4}$:

$$\begin{aligned} \text{LHS} &= 2x = 2(\tfrac{1}{4}) = \tfrac{1}{2} \\ \text{RHS} &= \sqrt{7-27x} = \sqrt{7-27(\tfrac{1}{4})} = \sqrt{\tfrac{28}{4} - \tfrac{27}{4}} \\ &= \sqrt{\tfrac{28-27}{4}} = \sqrt{\tfrac{1}{4}} = \tfrac{\sqrt{1}}{\sqrt{4}} = \tfrac{1}{2}, \end{aligned}$$

so the left- and right-hand sides are equal with $x = \frac{1}{4}$. Now what about $x = -7$?

$$\begin{aligned} \text{LHS} &= 2x = 2(-7) = -14 \\ \text{RHS} &= \sqrt{7-27x} = \sqrt{7-27(-7)} = \sqrt{7+189} = \sqrt{196} = 14, \end{aligned}$$

Here LHS $\neq$ RHS! What's going on?

Well, strictly speaking we should always check the answers when we conclude an OR statement. For example, if we declare that $z = 2$, then it follows that $z \cdot z = 2 \cdot 2$, i.e., $z^2 = 4$. But from this, by lemma 3.5, we conclude that

$$z = 2 \quad \text{or} \quad z = -2,$$

which is correct: we do indeed have that "$z = 2 \quad \text{OR} \quad z = -2$" is true—and this does not mean that both give meaningful answers to what we started with, only that at least one of them does. (In fact, this is similar to how we obtained $2 = 0$ in example (v).)

The reason we didn't bother checking the other equations is because if we were to carefully check the directions of our implications ($\Rightarrow$ or $\Leftarrow$), each of them could go both ways ($\Leftrightarrow$). For example,

$$x^2 - 5x + 6 = 0 \iff (x-2)(x-3) = 0 \iff x = 2 \quad \text{or} \quad x = 3.$$

Thus x being 2 or 3 is equivalent to $x^2 - 5x + 6$ equalling zero. On the other hand, in our $z = 2$ case, we have $z = 2 \implies z^2 = 4$, but not $z = 2 \impliedby z^2 = 4$. Indeed, if $z = -2$, it is true that $z^2 = 4$, but we cannot conclude that $z = 2$.

Similarly in the equation we solved, it is the first step, where we do

$$2x = \sqrt{7 - 27x} \implies (2x)^2 = 7 - 27x$$

which is a strictly one-sided implication. In fact, if $(2x)^2 = 7 - 27x$, we know by lemma 3.5 that what we can say is $2x = \pm\sqrt{7 - 27x}$, which is consistent with the LHS $= -14$ and RHS $= 14$ situation. Indeed, had we started with the equation $-2x = \sqrt{7 - 27x}$ instead, then applying the definition of square root (which we could see as *squaring both sides*) will result in the same equation $4x^2 + 27x - 7 = 0$. Thus information about the sign of the left-hand/right-hand sides is lost when we square both sides; we only retain equality up to sign.

Thus in general, when we "square both sides" of an equation, information about the sign is lost, so extra solutions (corresponding to LHS $= -$RHS) are introduced. Extra care should be taken when performing this operation—check your answers!

Exercise 3.10. 1. Write the following quadratic expressions in the form $a(x+p)^2 + q$.

a) $x^2 - 5x + 6$ b) $x^2 - 3x$ c) $5x^2 + x + 1$
d) $2x^2 + 1$ e) $x^2 - \sqrt{2}x + \sqrt{3}$ f) $17x^2 + 34x + 15$
g) $x^2 + x\sqrt{2} + x\sqrt{3}$ h) $x^2 + kx^2 + x + k$ i) $7 - 3x - 6x^2$

2. Consider the quadratic $\phi(x) = 9 - 3x - 5x^2$.

   a) Express ϕ in the form $s - 5(x+t)^2$, where $s, t \in \mathbb{Q}$.
   b) Explain why $\phi(x)$ is never more than $\frac{189}{20}$.
   c) Is this value attained by ϕ for some x?
   d) Does ϕ have a lower-bound as well, or only an upper-bound? Explain why.

3. Solve the following equations by factorising.

   a) $x^2+7x+12=0$ b) $x^2-3x-4=0$

   c) $x^2+5x+6=0$ d) $10x^2+x=21$

   e) $x^2+4x-10=2x+5$ f) $x^2-16=0$

   g) $x^2+3x=x$ h) $x^4-26x^2+25=0$

   i) $\sqrt{2x-1}=x$ j) $\sqrt{9-5x^2}=2x$

4. Solve the following equations.

   a) $x^2-8x-48=0$ b) $x^2+2x-48=-6$

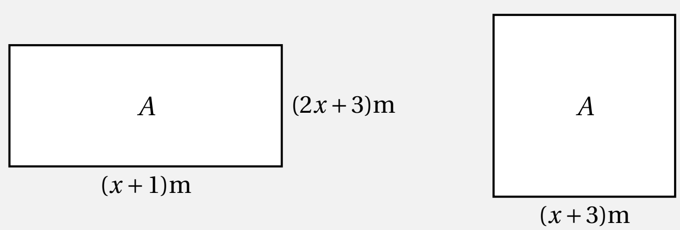
   c) $5x^2-21=10x$ d) $1=\sqrt{7x-x^2}$

   e) $x^2+13x+22=7$ f) $x^2-9x-39=-9$

   g) $2x^2+12x+10=0$ h) $5x^2+19x-68=-2$

   i) $3x^2+20x+36=4$ j) $19x+x^5=x+10x^3$

5. A rectangle and a square have the same area. Their dimensions are shown in metres below.



   Find the area A.

6. A rectangular field with an area of 75 m^2 is enclosed by a wooden fence. One side of the fence is 3 m longer than its adjacent side. What are the dimensions of the fence? Give answers accurate to 2 d.p.s.

7. Two boats start sailing from the same point. One travels north at 25 km/h. Two hours later the second boat starts travelling east at 20 km/h. How much time must pass from the departure of the first boat for them to be exactly 300 km apart?

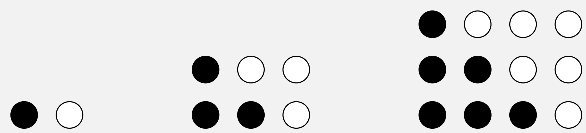
8. (The Quadratic Formula). Use completing the square to show that if $a \neq 0$, then

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right].$$

Hence deduce that if $b^2 - 4ac \geqslant 0$, the solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

9. Consider the three arrays below.



   a) How many dots are there in the fourth array of this pattern? How many of them are black?

   b) How many dots are there in the nth pattern? How many of them are black?

   c) Which array in this pattern contains 4950 black dots?

   d) In array 3, the number of black dots can be written $1 + 2 + 3$. Express the number of black dots in array 7 as a sum of integers in a similar way.

   e) What is the result of the sum $1 + 2 + 3 + \cdots + 999 + 1000$?

10. a) Solve the equation

$$\frac{1}{x} - \frac{1}{\sqrt{2}} = \frac{1}{x + \sqrt{2}},$$

giving your solution(s) in the form $p\sqrt{2} + q\sqrt{10}$ for $p, q \in \mathbb{Q}$.

   b) Solve the equation

$$\frac{1}{x + \sqrt{2} + \sqrt{3}} = \frac{1}{x} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}},$$

giving your answer in surd form. (MATSEC Sept '17)

11. Solve the following equations.

    a) $10x^2 + \frac{24}{x^2} = 31$

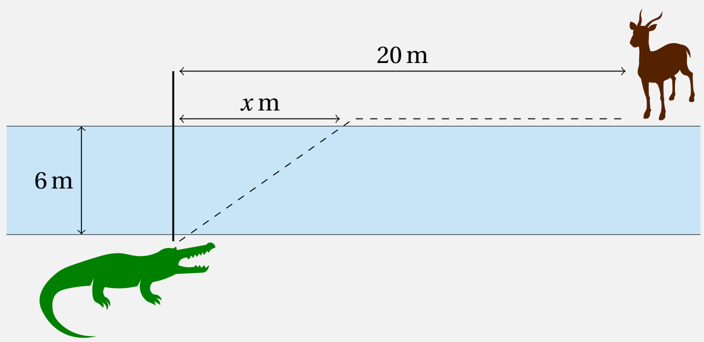
    b) $\sqrt{x+1} + \sqrt{2x+1} = \sqrt{3x+1}$

    c) $x + 2\sqrt{2} = 2\sqrt{x} + x\sqrt{2}$

    d) $3x^2 = \frac{32}{x^{4/3}} + 95x^{1/3}$

12. The cost of hiring a bus is €70. If nine of the seats are unoccupied, the cost per person is €1 more than each person would have to pay if all the seats were full. How many seats are there on the bus?

13. A crocodile is stalking a gazelle that is 20 m upstream on the opposite side of a river. Crocodiles travel at different speeds on land and in water. In water, they travel at 5 m/s, whereas on land, they travel at 4 m/s. Suppose the crocodile swims to a point that is x m upstream on the opposite bank of the river, and runs on land the rest of the way, as depicted below.



    a) Show that the time taken for the crocodile to reach the gazelle is given by

    $$T(x) = \frac{\sqrt{36+x^2}}{5} + \frac{20-x}{4}.$$

    b) Calculate the time taken if the crocodile does not travel on land.

    c) Calculate the time taken if the crocodile swims the shortest distance possible.

    d) In reality, the crocodile took five and a half seconds to get the gazelle. How far upstream did the crocodile get before it continued on land? Write your answer in the form $p + q\sqrt{106}$, where $p, q \in \mathbb{Q}$.

THEORY OF QUADRATIC EQUATIONS

Not all quadratics have roots. For instance, if we try to solve the equation

$$x^2 - 2x + 2 = 0,$$

completing the square yields

$$(x-1)^2 + 1 = 0 \implies (x-1)^2 = -1,$$

which cannot have any real solutions because of proposition 2.2.

In this section, we will try to demystify the relationship between quadratics' coefficients and their roots. In exercise 3.10(6), the number $b^2 - 4ac$ cropped up. This important constant can tell us a lot about the corresponding quadratic.

Definition 3.11 (Discriminant)**.** Let $\phi(x) = ax^2 + bx + c$ be a quadratic. The *discriminant* of ϕ, denoted $\Delta(\phi)$ or Δ, is the real number defined by

$$\Delta(\phi) = b^2 - 4ac.$$

Theorem 3.12. *Let $\phi(x) = ax^2 + bx + c$ be a quadratic, and let Δ be its discriminant.*

i) If $\Delta > 0$, then ϕ has two distinct real roots, given by $\frac{-b+\sqrt{\Delta}}{2a}$ and $\frac{-b-\sqrt{\Delta}}{2a}$.

ii) If $\Delta = 0$, then ϕ has one real root called a repeated root, *given by $-\frac{b}{2a}$.*

iii) If $\Delta < 0$, then ϕ has no real roots.

Furthermore, if $a > 0$ then $\phi(x) > 0$ for all $x \in \mathbb{R}$, whereas if $a < 0$, then $\phi(x) < 0$ for all $x \in \mathbb{R}$.

Proof. In exercise 3.10(6), we saw that by completing the square, we can rewrite $\phi(x) = ax^2 + bx + c$ as

$$a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{-\Delta}{4a^2}\right].$$

Assuming for now that $\Delta \geqslant 0$, we have

$$\begin{aligned}\phi(x) = 0 &\iff a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{-\Delta}{4a^2}\right] = 0 \\ &\iff \left(x + \frac{b}{2a}\right)^2 + \frac{-\Delta}{4a^2} = 0\end{aligned}$$

$$\begin{aligned}
&\iff \left(x+\frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2} \\
&\iff x+\frac{b}{2a} = \pm\frac{\sqrt{\Delta}}{2a} \qquad \text{(by lemma 3.5, since } \Delta \geqslant 0) \\
&\iff x = \frac{-b\pm\sqrt{\Delta}}{2a}.
\end{aligned}$$

Now, if $\Delta > 0$, then the two solutions are distinct, since they differ by the non-zero quantity $\frac{\sqrt{\Delta}}{a}$.

If $\Delta = 0$, then they are both equal to $\frac{-b\pm\sqrt{0}}{2a} = -\frac{b}{2a}$, so we have one "repeated" solution.

Finally, if $\Delta < 0$, we can again rewrite the equation $\phi(x) = 0$ as

$$\left(x+\frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2},$$

just as we did above. This is equivalent to the original equation since it was deduced using two-sided implications ($\Leftrightarrow$). Now since $\Delta < 0$, then $\frac{\Delta}{4a^2} < 0$. But the left-hand side is non-negative independently of x, since no number squared can be negative (proposition 2.2). Thus this equation can have no solutions.

Moreover, we have

$$\phi(x) = ax^2+bx+c = a\underbrace{\left[\left(x+\frac{b}{2a}\right)^2 + \underbrace{\frac{-\Delta}{4a^2}}_{>0}\right]}_{\text{denote this by } K(x)},$$

i.e., $\phi(x) = aK(x)$ where $K(x)$ is always a positive number independently of x. Therefore it follows that if $a > 0$, $\phi(x) > 0$, and similarly if $a < 0$, $\phi(x) < 0$. □

Examples 3.13. We give some examples of applications of this theorem.

i) x^2-5x+5 has real and distinct roots since $\Delta = 5^2-4(1)(5) = 5 > 0$.

ii) x^2-6x+9 has a repeated root, since $\Delta = 6^2-4(1)(9) = 0$.

iii) $3x^2-5x+11$ has no real roots, since $\Delta = 5^2-4(3)(11) = -107 < 0$. Furthermore, since $a = 3 > 0$, then this quadratic is always positive for any value of x.

iv) $x^2+ax+3a^2$ where a is a non-zero constant has no real roots, since $\Delta = a^2-4(1)(3a^2) = -11a^2 < 0$ for all non-zero a.

v) We determine which values of b make $3x^2 + bx + 3$ have repeated roots. This happens when the discriminant $\Delta = b^2 - 4(3)(3) = 0$, that is, when $b^2 - 36 = 0$, that is, when $b = \pm 6$.

vi) We prove that $9x^2 - 12ax + 4a^2$ always has repeated roots. Indeed, we have $\Delta = (-12a)^2 - 4(9)(4a^2) = 144a^2 - 144a^2 = 0$, no matter the value of a.

Now that we know when we a quadratic has solutions, we explore the question of when factorisation is possible. So far, we have seen that when a quadratic factorises, it tends to looks something like $(x - \alpha)(x - \beta)$, where α and β are its roots. For instance, $x^2 - 5x + 6$ becomes $(x - 2)(x - 3)$, where 2 and 3 are its roots.

This makes sense: quadratics are algebraic expressions which are (typically) zero at two instances. Expressions like $x - 2$ and $x - 3$ are zero at one instance (namely when $x = 2$ and 3 respectively), so if we want to create something that is zero at both instances, we multiply them to get

$$(x-2)(x-3) = x^2 - 5x + 6,$$

which is an expression equal to zero at both $x = 2$ and $x = 3$. Similarly if we want to create an expression which is zero when x is $2, 3$ or 4, we can do

$$(x-2)(x-3)(x-4) = x^3 - 8x^2 + 26x - 24,$$

which is zero at all three, but then we will be entering the realm of *cubics*, so we won't go there yet.

This train of thought gets interesting when we think about roots which we find, not by factorisation, but by completing the square when we aren't able to factorise. For instance, the roots of $7x^2 - 5x - 6$ in examples 3.8(ii) were $\frac{5\pm\sqrt{193}}{14}$. So what happens if we expand the product $\big(x - \frac{5+\sqrt{193}}{14}\big)\big(x - \frac{5-\sqrt{193}}{14}\big)$? This expression is also zero at both those instances of x, so will we get the quadratic we started with when we expand? Or does that only happen for quadratics we manage to solve by factorising? Let's see:

$$\begin{aligned}
&\Big(x - \frac{5+\sqrt{193}}{14}\Big)\Big(x - \frac{5-\sqrt{193}}{14}\Big) \\
&= x^2 - \frac{(5+\sqrt{193})x + (5-\sqrt{193})x}{14} + \Big(\frac{5+\sqrt{193}}{14}\Big)\Big(\frac{5-\sqrt{193}}{14}\Big) \\
&= x^2 - \frac{10}{14}x + \frac{25-193}{14^2} = x^2 - \tfrac{5}{7}x - \tfrac{6}{7},
\end{aligned}$$

which is the original quadratic, simply scaled down by a factor of $\frac{1}{7}$! So it seems the answer to this question is yes, that in fact, all quadratics are equal to the product $(x-\alpha)(x-\beta)$ (possibly scaled up when the coefficient of x^2 isn't 1), where α and β are its roots:

Theorem 3.14. *Let $\phi(x) = ax^2 + bx + c$ be a quadratic, and let Δ be its discriminant. If $\Delta \geqslant 0$, then $\phi(x)$ can expressed in the form*

$$\phi(x) = a(x-\alpha)(x-\beta),$$

where $\alpha, \beta \in \mathbb{R}$ are its roots. Furthermore, the quadratic $K(x-\alpha)(x-\beta)$ has the same roots as ϕ for any non-zero real number K.

Proof. The roots of the quadratic $\phi(x) = ax^2 + bx + c$ with $\Delta \geqslant 0$ are $x = \frac{-b\pm\sqrt{\Delta}}{2a}$ (theorem 3.12). Denote these by α, β. Now

$$\begin{aligned}
\phi(x) &= ax^2 + bx + c \\
&= a\big(x^2 + \tfrac{b}{a}x + \tfrac{c}{a}\big) \\
&= a\Big(\big(x + \tfrac{b}{2a}\big)^2 - \big(\tfrac{b}{2a}\big)^2 + \tfrac{c}{a}\Big) \\
&= a\Big(\big(x + \tfrac{b}{2a}\big)^2 - \tfrac{b^2}{4a^2} + \tfrac{c}{a}\Big) \\
&= a\Big(\big(x + \tfrac{b}{2a}\big)^2 - \tfrac{b^2+4ac}{4a^2}\Big) \\
&= a\Big(\big(x + \tfrac{b}{2a}\big)^2 - \tfrac{\Delta}{4a^2}\Big) \\
&= a\Big(\big(x + \tfrac{b}{2a}\big)^2 - \big(\tfrac{\sqrt{\Delta}}{2a}\big)^2\Big) \\
&= a\Big(\big(x + \tfrac{b}{2a} + \tfrac{\sqrt{\Delta}}{2a}\big)\big(x + \tfrac{b}{2a} - \tfrac{\sqrt{\Delta}}{2a}\big)\Big) \qquad \text{(lemma 2.5)} \\
&= a\big(x - \big(\tfrac{-b-\sqrt{\Delta}}{2a}\big)\big)\big(x - \big(\tfrac{-b+\sqrt{\Delta}}{2a}\big)\big) \\
&= a(x-\alpha)(x-\beta),
\end{aligned}$$

as required. Now to show that $K(x-\alpha)(x-\beta)$ has the same roots for non-zero K, observe that

$$K(x-\alpha)(x-\beta) = 0 \iff K = 0 \quad \text{or} \quad x - \alpha = 0 \quad \text{or} \quad x - \beta = 0,$$

and since $K \neq 0$, the result follows. □

Example 3.15. The quadratic $2x^2 - 4x - 5$ has roots $(2 \pm \sqrt{14})/2$. Indeed,

$$\begin{aligned}
&2\Big(x - \frac{2+\sqrt{14}}{2}\Big)\Big(x - \frac{2-\sqrt{14}}{2}\Big) \\
&= 2x^2 - (2+\sqrt{14})x - (2-\sqrt{14})x + 2\Big(\frac{2+\sqrt{14}}{2}\Big)\Big(\frac{2-\sqrt{14}}{2}\Big) \\
&= 2x^2 - 4x + \frac{2^2 - 14}{2} = 2x^2 - 4x - 5.
\end{aligned}$$

This theorem is very useful as we shall soon discover, but we still haven't answered the question of when factorising a given quadratic is possible in the usual sense. For that, we have the following theorem.

Theorem 3.16 (Quadratic Factorisation)**.** *Let $\phi(x) = ax^2 + bx + c$ be a quadratic with $\Delta \geqslant 0$, and suppose $a, b, c \in \mathbb{Z}$. Then $\Delta = k^2$ for some $k \in \mathbb{Z}$ if and only if we can write*

$$\phi(x) = (Ax - B)(Cx - D)$$

for $A, B, C, D \in \mathbb{Z}$.

Example 3.17. For example, $6x^2 + x - 2$ can be written as $(2x-1)(3x+2)$. Indeed, $\Delta = 1^2 - 4(6)(-2) = 49 = 7^2 \in \{k^2 : k \in \mathbb{Z}\}$.

Thus if you are struggling to determine the factorisation of a quadratic, evaluate its discriminant and check if it is a square number. If it isn't, you can stop trying and use completing the square.

The proof of theorem 3.16 is quite interesting, it involves some number theoretic ideas in addition to just usual algebra.

Proof of theorem 3.16. If $\Delta = k^2$, then

$$k^2 = b^2 - 4ac \implies b^2 - k^2 = 4ac \implies (b-k)(b+k) = 4ac,$$

so one of $b-k$ and $b+k$ is even. Moreover, both are even, since they differ by the even number $2k$. Thus we can pick two even numbers s and t such that $st = 4a$, where s divides $b-k$ and t divides $b+k$.

Thus by theorems 3.12 and 3.14,

$$\begin{aligned}\phi(x) &= a\Big(x - \frac{-b-k}{2a}\Big)\Big(x - \frac{-b+k}{2a}\Big)\\ &= \frac{st}{4}\Big(x - \frac{-b-k}{st/2}\Big)\Big(x - \frac{-b+k}{st/2}\Big)\\ &= \Big(\frac{s}{2}x - \frac{-b-k}{t}\Big)\Big(\frac{t}{2}x - \frac{-b+k}{s}\Big),\end{aligned}$$

where we can see that $\frac{s}{2}, \frac{t}{2}, \frac{-b-k}{t}, \frac{-b+k}{s} \in \mathbb{Z}$, as required.

Conversely, if we can write $\phi(x)$ as $(Ax-B)(Cx-D)$, then one of its roots is $\frac{B}{A}$, so (w.l.o.g.), we have

$$\frac{-b+\sqrt{\Delta}}{2a} = \frac{B}{A} \implies \Delta = \Big(\frac{2aB}{A} + b\Big)^2,$$

so Δ is a square, provided that $\frac{2aB}{A}+b$ is an integer. Clearly $\frac{2aB}{A}+b \in \mathbb{Q}$, so we may write it as $\frac{X}{Y}$ where X and Y have no common factors (otherwise we can just cancel them). But then the denominator of $\Delta = \frac{X^2}{Y^2}$ written in its lowest terms is Y^2, and since $\Delta = b^2-4ac \in \mathbb{Z}$, we must have $Y^2 = 1$ i.e., $Y = \pm 1$. Therefore $\frac{2aB}{A}+b = \pm X \in \mathbb{Z}$, which completes the proof. □

Corollary 3.18. *Let $\phi = ax^2+bx+c$ be a quadratic with integer coefficients, and suppose $\Delta \geqslant 0$. Then $\Delta = k^2$ for some $k \in \mathbb{Z}$ if and only if the roots of ϕ are rational.*

Proof. If $\Delta = k^2$, then we may write $\phi(x)$ as $(Ax-B)(Cx-D)$ where $A,B,C,D \in \mathbb{Z}$, so its roots are $\frac{B}{A}$ and $\frac{D}{C}$ which are clearly rational.

Conversely, if the roots of ϕ are rational, let $\frac{B}{A}$ denote one of them. Then we have (w.l.o.g.) that $\frac{-b+\sqrt{\Delta}}{2a} = \frac{B}{A}$, and the proof continues identically to that of the converse in theorem 3.16. □

Don't confuse what this theorem is telling us with what theorem 3.14 is saying. What we have is that, in theory, any quadratic with $\Delta \geqslant 0$ can be factorised as $a(x-\alpha)(x-\beta)$, but due to our human limitations, we can't immediately notice the factors unless they are sufficiently "nice" (i.e., rational, as in corollary 3.18). So *in theory*, you should be able to look at

$$x^2 - x - 1 = 0$$

and say "of course! This factorises as

$$\left(x - \tfrac{1+\sqrt{5}}{2}\right)\left(x - \tfrac{1-\sqrt{5}}{2}\right) = 0."$$

But in practice, we can only manage what theorem 3.16 tells us (by sight).

We will conclude this section by discussing the relationship between the roots of a quadratic and its coefficients. This leads us to the following famous relations.

Theorem 3.19 (Viète's Formulæ for Quadratics)**.** *Let $\phi(x) = ax^2+bx+c$ be a quadratic. Then*

$$\phi(x) = a(x^2 - (\alpha+\beta)x + \alpha\beta) = a(x^2 - \Sigma x + \Pi),$$

where α, β are its roots, and Σ and Π denote the sum and product of the roots respectively. In other words, $\alpha+\beta = -{}^{b}/_{a}$ and $\alpha\beta = {}^{c}/_{a}$.

Proof. By theorem 3.12, ϕ can be expressed as $a(x-\alpha)(x-\beta)$, which when expanded, gives

$$\phi(x) = a(x^2 - (\alpha+\beta)x + \alpha\beta) = ax^2 - a(\alpha+\beta)x + a\alpha\beta.$$

Comparing this with ax^2+bx+c yields $\alpha+\beta = -{}^b/_a$ and $\alpha\beta = {}^c/_a$, as required.[6] □

Example 3.20. If you want to "create" an equation with solutions $x=2$ and $x=5$ (say), you now have two ways. The first is to simplify $K(x-2)(x-5)$, where you can choose $K \neq 0$ to be any number you like (you could just take $K=1$).

Alternatively, by Viète's formulæ (theorem 3.19), we have the sum $\Sigma = 2+5 = 7$ and the product $\Pi = 2\cdot 5 = 10$. Thus $a(x^2-7x+10)$ for any $a \neq 0$ has the required roots.

Example 3.21. Viète's formluæ also allow us to modify roots of quadratics without having to find them. This is not why they are important, but doing this requires you to have a firm understanding of what the theorem is actually saying (and is therefore something which examination boards like to ask of you!). Suppose we have the quadratic $x^2-17x+15$ whose roots are α and β. Can we devise a quadratic whose roots are, for example, $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$? A naïve way of doing this is to actually find the roots of the quadratic (α and β), then determine $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ explicitly, and then simplify the expression $\left(x-\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)$, which by theorem 3.14 has the required roots.

Naïve solution: By completing the square, we get that $x^2-17x+15 = (x-\frac{17}{2})^2 - \frac{289}{4} + 15 = (x-\frac{17}{2})^2 - \frac{229}{4}$. Thus the roots are given by $x^2-17x+15 = 0 \implies (x-\frac{17}{2})^2 = \frac{229}{4} \implies x = \frac{17\pm\sqrt{229}}{2}$. Therefore

$$\frac{\alpha}{\beta} = \frac{\frac{17+\sqrt{229}}{2}}{\frac{17-\sqrt{229}}{2}} = \frac{17+\sqrt{229}}{17-\sqrt{229}}, \qquad \text{and} \qquad \frac{\beta}{\alpha} = \frac{1}{\alpha/\beta} = \frac{17-\sqrt{229}}{17+\sqrt{229}},$$

so the required quadratic is given by

$$\begin{aligned}&\left(x - \frac{17+\sqrt{229}}{17-\sqrt{229}}\right)\left(x - \frac{17-\sqrt{229}}{17+\sqrt{229}}\right)\\ &= x^2 - \frac{17+\sqrt{229}}{17-\sqrt{229}}x - \frac{17-\sqrt{229}}{17+\sqrt{229}}x + \left(\frac{17+\sqrt{229}}{17-\sqrt{229}}\right)\left(\frac{17-\sqrt{229}}{17+\sqrt{229}}\right)\end{aligned}$$

[6] The "comparing coefficients" technique is justified in proposition 5.36 later on. The basic idea is that, if two quadratics are equal for any value of x, it follows that their coefficients must be equal.

$$
\begin{aligned}
&= x^2 - \frac{(17+\sqrt{229})^2 + (17-\sqrt{229})^2}{(17+\sqrt{229})(17-\sqrt{229})}x + 1 \\
&= x^2 - \frac{289 + 34\sqrt{229} + 229 + 289 - 34\sqrt{229} + 229}{289 - 229}x + 1 \\
&= x^2 - \frac{259}{15}x + 1,
\end{aligned}
$$

which we can give as $15x^2 - 259x + 15$, since this has the same roots by theorem 3.14.

Now we proceed to give a simpler solution using theorem 3.19.

Solution using theorem 3.19: By the theorem, we have $\Sigma = \alpha + \beta = 17$, and $\Pi = \alpha\beta = 15$. Now since the required expression is also a quadratic, then it will be of the same form, that is, $x^2 - \Sigma x + \Pi$, where the new sum (Σ_N) and product (Π_N) are $\Sigma_N = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{17^2 - 2(15)}{15} = \frac{259}{15}$, and $\Pi_N = \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right) = 1$. Hence the required quadratic is $x^2 - \Sigma_N x + \Pi_N = x^2 - \frac{259}{15}x + 1$, which has the same roots as $15x^2 - 259x + 15$.

Example 3.22. Here is another interesting application of Viète's formulæ. Suppose we have information about the roots of a given quadratic. Can we express this information as a relation in terms of the coefficients? This is precisely how we should think of Viète's formulæ: as a bridge between the coefficient world to the root world (and vice-versa). For instance, suppose we know that $ax^2 + bx + c$ has one root equal to twice the other. What conditions does this force on a, b and c?

Well, we can let the roots equal α and 2α (rather than just α and β), this way we incorporate the information we know about them. Then Viète's formulæ become

$$\alpha + 2\alpha = -\frac{b}{a} \qquad \text{and} \qquad \alpha(2\alpha) = \frac{c}{a}.$$

We don't care about α here, we want to obtain a relation about a, b and c. Since we have two equations, we can use one to eliminate α from the other. The first one tells us that $\alpha = -\frac{b}{3a}$, so we can just plug this into the second one and get

$$2\left(-\frac{b}{3a}\right)^2 = \frac{c}{a} \implies 2b^2 - 9ac = 0.$$

Thus having one root equal to double the other implies that $2b^2 - 9ac = 0$. It would be nice if this is also a sufficient condition, i.e., if the converse is also true, so that we get an equivalence. Indeed, consider the quadratic $ax^2 + bx + c$ and suppose the coefficients satisfy $2b^2 - 9ac = 0$. Then by Viète's formulæ, we have

$$\alpha + \beta = -\frac{b}{a} \qquad \text{and} \qquad \alpha\beta = \frac{c}{a}.$$

We can square the first equation to get a b^2, which we can then substitute for a $\frac{9}{2}ac$ to incorporate our condition on the coefficients:

$$(\alpha+\beta)^2 = \frac{b^2}{a^2} \implies (\alpha+\beta)^2 = \frac{\frac{9}{2}ac}{a^2} = \frac{9c}{2a}.$$

Now we should aim to get an equation involving only α and β, because what we want to end up showing is that our condition imposes that one root is double the other. Notice the presence of a $\frac{c}{a}$ on the RHS of our equation, we can substitute this with $\alpha\beta$ by the second Viète formula to get

$$\begin{aligned}(\alpha+\beta)^2 = \tfrac{9}{2}\alpha\beta &\implies 2\alpha^2 - 5\alpha\beta + 2\beta^2 = 0\\ &\implies (2\alpha-\beta)(\alpha-2\beta) = 0\\ &\implies \alpha = 2\beta \quad \text{or} \quad \beta = 2\alpha,\end{aligned}$$

i.e., that one root is double the other. Thus we have shown that

$$\text{One root of } ax^2 + bx + c \text{ is double the other} \iff 2b^2 - 9ac = 0.$$

This is an interesting result, notice that $2b^2 - 9ac$ is quite similar to the familiar discriminant expression $b^2 - 4ac$. The latter is zero when the roots are the same, the former acts as a sort of modified discriminant which is zero when one root is twice the other.

Exercise 3.23. 1. Determine the nature of the roots of the following quadratic equations without solving them.

a) $x^2 - 2x = -5$ b) $x^2 - 5x + 9 = x$

c) $5x^2 - 5x + 1 = 0$ d) $k^2x^2 + kx + 4 = 0$, $k \neq 0$

e) $x^2 - kx + 2k = x + k$, $k \neq 1$ f) $kx^2 = k$, $k \neq 0$

2. For what value(s) of k do the following quadratics have repeated roots?

a) $x^2 + 2kx + 1$ b) $3x^2 + x + k$ c) $kx^2 + kx + 4$

3. Prove that the roots of the equation $4+b-bx-x^2=0$ are real and distinct for any $b \in \mathbb{R}$.

4. Consider the quadratic $\phi(x) = x^2 - 6x + 13$.

   a) Prove that ϕ has no real roots.

   b) Suppose that i is a special number with the property that $i^2 = -1$ (we know that there is no such real number, but pretend it exists anyway). Show that $3+2i$ and $3-2i$ are roots of the quadratic ϕ.

   c) Suppose the quadratic ax^2+bx+c has no real roots. Show that
   $$x = \frac{-b \pm i\sqrt{-\Delta}}{2a}$$
   are two roots of this quadratic, where i behaves as described in part (b).

5. Prove the following, and **memorise them well!**

   a) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

   b) $\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

6. Given the equation $2x^2+7x-3=0$ has roots α and β, form QEs whose roots are given by the following expressions:

   a) α^2, β^2 b) α^3, β^3 c) $\frac{2}{\alpha}, \frac{2}{\beta}$

   d) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ e) $\alpha^3-1,\ \beta^3-1$ f) $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}$

   g) $\frac{\alpha^2}{\beta}, \frac{\beta^2}{\alpha}$ h) $\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\beta+\alpha}$ i) $\alpha+3\beta, \beta+3\alpha$

7. The equation $x^2+(2r-3)x+1=0$ has the roots α and β. Form an equation whose roots are $\frac{\alpha}{\alpha\beta+1}$ and $\frac{\beta}{\alpha\beta+1}$.

8. One of the roots of the equation $x^2+px+8=0$ is the square of the other. Find p.

9. The equation $rx^2+p=x+1$ has one root double the other. Show that $2=9r(p-1)$.

10. If α and β are the roots of $x^2 - 2kx + k - 2 = 0$, form the quadratic equation whose roots are $\frac{\alpha^2}{\beta} - 1$, $\frac{\beta^2}{\alpha} - 1$.

11. Given that the equation $ax^2 + bx + c = 0$ has roots α and β, form QEs whose roots are given by:

    a) $a\alpha, a\beta$

    b) $\dfrac{b\alpha}{\beta^2}, \dfrac{b\beta}{\alpha^2}$

    c) $\dfrac{1}{\alpha - c}, \dfrac{1}{\beta - c}$

12. If α^2 and β^2 are the roots of the equation $x^2 - 10x + 9 = 0$, where $\alpha, \beta > 0$, evaluate $\alpha\beta$ and $\alpha + \beta$. Hence, write down the equation whose roots are α and β.

13. Show that if a and b are both positive or both negative, then

$$\frac{x}{x-a} + \frac{x}{x-b} = 1$$

has two distinct real solutions.

IV. LOGARITHMS

WHAT if, instead of studying expressions like x^2, we swap things around and consider 2^x, where now the index itself is an unknown? If we have the equation $3^x = 27$, we know that $x = 3$ is a solution, but can we develop a general understanding of such equations without cherry-picked examples? Is there a real number x such that, $3^x = 28$ instead? (for instance).

Definition 4.1 (Logarithm)**.** Let $a, x \in (0, \infty)$ such that $a \neq 1$. The *logarithm of x to base a*, denoted

$$\log_a x,$$

is a number $y \in \mathbb{R}$ such that $a^y = x$.

Example 4.2. For example, $\log_2 8 = 3$, and $\log_5 5 = 1$. Indeed, $\log_x x = 1$ for any $x \in (0, \infty)$.

We essentially have three ways to denote the same relationship between three numbers. Taking the example of 2, 3 and 8, we have

$$2^3 = 8, \qquad \sqrt[3]{8} = 2, \qquad \text{and} \qquad \log_2 8 = 3.$$

In each case, we have a different number as subject of the equation. The logarithm allows us to "access" the power directly, making it subject.

Theorem 4.3. *Let $a, x \in (0, \infty)$ where $a \neq 1$. Then $\log_a x$ exists, and moreover it is unique, that is, there exists only one real number y such that $a^y = x$.*

Proving the existence and uniqueness of $\log_a x$ in $\mathbb{R}$ requires a lot of work (analysis) which we will not bother getting into here, but we might revisit it later.

Examples 4.4. It's important to get a feel for logarithms. For instance, what should $\log_3 10$ be? We know that 3^2 is 9, and 3^3 is 27, so $\log_3 10$ should just a bit more than 2, probably around 2.1. In fact, $\log_3 10 \approx 2.095$.

Similarly, what is $\log_2 56$? We know $2^5 = 32$ and $2^6 = 64$, so we'd expect $\log_2 56$ to be between 5 and 6, closer 6. We can hazard a guess of 5.7. In fact, the real value is around 5.833.

What about $\log_{100} 5$ be? We know that $\sqrt{100} = 100^{1/2} = 10$, and $100^0 = 1$, so we'd expect something between 0 and $1/2$. We could guess 0.3. The real answer is about 0.349.

One final example, what should $\log_6 \left(\frac{1}{3}\right)$ be? Well, $6^0 = 1$, and $6^{-1} = \frac{1}{6}$, so we'd expect something between the two, perhaps -0.5? The real answer is in fact -0.613.

Remark 4.5. $\log_a x$ only exists when $x > 0$. For instance, what should $\log_5(-4)$ equal? If we think about it, $5^1 = 5$, $5^2 = 25$, $5^3 = 125$, etc., so taking powers of 5 larger than 1 will just give us larger and larger positive numbers. Perhaps we consider powers between 0 and 1? Well things like $5^{1/2}$ give us $\sqrt{5} \approx 2.2$, so still a positive number. It's only once we get to 5^0 do we get 1. Then if we consider negative powers, they are simply reciprocals of the positive ones, so $5^{-1/2} = \frac{1}{\sqrt{5}} \approx 0.45$, $5^{-2} = \frac{1}{25} = 0.04$, $5^{-3} = \frac{1}{125} = 0.008$. Large negative powers bring us close to zero (e.g. $5^{-10} = \frac{1}{9765625} \approx 0.0000001024$), but we can't do better than that; we can't even get $5^x = 0$. Thus $\log_a x$ only makes sense when x is positive.

The following is another analytic property of logarithms, which we've already been assuming tacitly in our reasoning about them:

Theorem 4.6. *Let $a \in (0,\infty)$ where $a \neq 1$. Then* $\log_a x$ *is an* injection*, that is, if* $\log_a x = \log_a y$*, then* $x = y$*. Moreover,* $\log_a$ *is increasing, i.e.,*

$$x < y \implies \log_a x < \log_a y.$$

Later on when doing functions, we will talk more about injections. The function $f(x) = (x-1)^2$ is not an injection for example, since $f(3) = f(-1)$, but $3 \neq -1$. In general, just because two outputs of a function are the same, it doesn't mean the two inputs are the same. But for $\log_a$, it is true. (Notice a is fixed here!)

Thus if we come across the equation $\log_3(x+7) = \log_3(2x+5)$ (for instance), we can infer that $x+7 = 2x+5$, i.e., that $x = 2$.

Examples 4.7. We give two examples of applications of logarithms. First, we solve the equation $4^x = 8$. By the definition, the desired x is given by $\log_4 8$. Indeed, plugging this into a calculator gives $x = {}^3\!/_2$. Let us verify that this is the answer: $4^{3/2} = 4^{3\cdot\frac{1}{2}} = (4^3)^{1/2} = 64^{1/2} = \sqrt{64} = 8$, as required.

For the next example, suppose we place €2000 in a bank savings account, which offers 1.2% interest per annum. How many years must pass before the account balance exceeds €2500, assuming that no other deposits/withdrawals are made? Well, the first step here is to notice that the balance after a year is given by €2000 + 1.2%(€2000) = €2000(1 + 1.2%) = €2024. Similarly for the next year, we do €2024 + 1.2%(€2024) = €2024(1 + 1.2%) = €2048.29. But this is the same as doing €2000(1 + 1.2%)(1 + 1.2%). In fact for the following year, we would work out €2000(1 + 1.2%)(1 + 1.2%)(1 + 1.2%), and in general, after n years, the balance is €2000$(1 + 1.2\%)^n$.

So what we want is a value n such that €2000$(1 + 1.2\%)^n$ = €2500. This equation simplifies to $\left(\frac{253}{250}\right)^n = \frac{5}{4}$, so the required n is given by $\lceil \log_{\frac{253}{250}}\left(\frac{5}{4}\right)\rceil = 19$ years.[7]

By definition, logarithms only allow us to solve equations of the form $a^x = b$, so any seemingly more complicated equations must be reduced to ones of this form using techniques we already know about indices.

Examples 4.8. Let us give a series of increasingly more "complicated" equations to illustrate how each is reduced to the form $a^x = b$.

(i) $2^x = 256$
Nothing much to do here, by definition, the answer is $x = \log_2(256) = 8$.

(ii) $2^x \cdot 3^x = 256$.
Here we can't take a logarithm yet since the equation is not of the desired form, but we can use law IV from theorem 2.22 to rewrite the equation as $(2\cdot 3)^x = 256$, i.e., $6^x = 256$. Therefore the solution is $x = \log_6 256$.

This time we do not get a rational answer, so we leave it in the exact form $x = \log_6 256$. We can get a sense of how large this number is by observing that $6^3 = 216$ and $6^4 = 1296$, so we expect it to be around 3.1 (it's actually ≈ 3.09).

(iii) $2^x \cdot 3^{x+1} = 256$.
We can't combine the powers this time, but we can use law I of indices to split the $x+1$, giving $2^x \cdot 3^x \cdot 3^1 = 256$. Moving the 3 to the other side, the equation becomes $2^x \cdot 3^x = \frac{256}{3}$. Proceeding as in (ii), we have

$$(2\cdot 3)^x = \frac{256}{3} \implies x = \log_6\left(\frac{256}{3}\right).$$

(iv) $2^{x+1} \cdot 3^{2x+1} = 256$.
Proceeding as before, we first split the exponents to get $2^x \cdot 2^1 \cdot 3^{2x} \cdot 3^1 = 256$. This time we have 3^{2x} rather than 3^x, but we can use law III to write it as $(3^2)^x$ instead, i.e., as 9^x, so that the equation becomes $2^x \cdot 2^1 \cdot 9^x \cdot 3^1 = 256$. Placing numbers on one side and exponentials on the other gives

$$2^x \cdot 9^x = \frac{256}{2\cdot 3} \implies 18^x = \frac{128}{3} \implies x = \log_{18}\left(\frac{128}{3}\right).$$

(v) $9^{3x} \cdot 3^{1+x} = 15^{2x+3}$.
This final example demonstrates the general strategy we've developed:

$$9^{3x} \cdot 3^{1+x} = 15^{2x+3}$$

[7] $\lceil x \rceil$ denotes the *ceiling* of x, i.e., the smallest integer larger than x. Here we are using it since presumably, the interest is computed at the end of the year. Since 18 years are not sufficient ($\log_{\frac{253}{250}}\left(\frac{5}{4}\right) \approx 18.71$), we round up to 19 years.

$$\begin{aligned}
\implies \quad & (9^3)^x \cdot 3^1 \cdot 3^x = 15^{2x} \cdot 15^3 \\
\implies \quad & \frac{(9^3)^x \cdot 3^x}{(15^2)^x} = \frac{15^3}{3} \\
\implies \quad & \frac{(9^3 \cdot 3)^x}{(15^2)^x} = 1125 \\
\implies \quad & \left(\frac{9^3 \cdot 3}{15^2}\right)^x = 1125 \\
\implies \quad & \left(\frac{243}{25}\right)^x = 1125 \\
\implies \quad & x = \log_{\frac{243}{25}} 1125.
\end{aligned}$$

Examples 4.9. Here we give two examples of a different kind.

(i) $3^{2x} + 6 = 5 \cdot 3^x$.
In contrast to the equations in examples 4.8, here we have three terms, so we can't just divide and multiply as easily. The trick is to rearrange this equation as

$$3^{2x} - 5 \cdot 3^x + 6 = 0 \implies (3^x)^2 - 5 \cdot (3^x) + 6 = 0.$$

What we have here is the QE $t^2 - 5t + 6 = 0$ with $t = 3^x$. Thus we can factorise to get

$$\begin{aligned}
& (3^x - 2)(3^x - 3) = 0 \\
\implies \quad & 3^x = 2 \quad \text{or} \quad 3^x = 3 \\
\implies \quad & x = \log_3 2 \quad \text{or} \quad x = 1,
\end{aligned}$$

which are both valid solutions.

(ii) $4^{x+1} + 2^x = 1$.
Just as in (i), we notice that the equation has more than two terms, so we need to try a similar strategy. This time the quadratic is more well-hidden, but it's in there:

$$4^{x+1} + 2^x = 1 \implies 4 \cdot 4^x + 2^x = 1 \implies 4 \cdot (2^x)^2 + 2^x - 1 = 0.$$

We have $4t^2 + t - 1 = 0$ with $t = 2^x$. Proceeding by completing the square, we get

$$(2^x + \tfrac{1}{8})^2 - \tfrac{17}{64} = 0 \implies 2^x = \tfrac{-1 \pm \sqrt{17}}{8} \implies x = \log_2(\tfrac{-1+\sqrt{17}}{8}).$$

Notice that we discard the solution corresponding to $2^x = \frac{-1-\sqrt{17}}{8}$ since $\frac{-1-\sqrt{17}}{8} < 0$ (remark 4.5).

THE LAWS OF LOGARITHMS

Some indicial equations present multiple courses of action. For instance, given the equation $2^{3x-1} = 5$, we could either split up the index, in accordance with the strategy we were using in examples 4.8, to get

$$2^{3x-1} = 5 \implies (2^3)^x \cdot 2^{-1} = 5 \implies 8^x = 10 \implies x = \log_8 10,$$

or else we could invoke the definition of logarithm immediately to get

$$2^{3x-1} = 5 \implies 3x - 1 = \log_2 5 \implies x = \tfrac{1}{3}(\log_2 5 + 1).$$

These answers look wildly different, but by theorem 4.3, we expect equations of this form to have a unique solution; so we expect that

$$\log_8 10 = \tfrac{1}{3}(\log_2 5 + 1),$$

which is true, in fact. But how can we see that this is true without the context of the equation we started with? In this section, we aim to establish some laws of logarithms so that we may better understand equalities of logarithmic quantities such as the one above.

The key thing to remember about logarithms is that what they give us is an index. The following theorem is essentially a translation of laws I–III in theorem 2.22, in terms of logarithms.

Theorem 4.10 (Laws of Logarithms)**.** *Let $a, x, y \in (0, \infty)$ such that $a \neq 1$, and let $c \in \mathbb{R}$. Then we have the following laws.*

I) $\log_a x + \log_a y = \log_a (xy)$

II) $\log_a x - \log_a y = \log_a \left(\dfrac{x}{y}\right)$

III) $c \log_a x = \log_a (x^c)$

Proof. Let $u = \log_a x$ and $v = \log_a y$. By definition of $\log_a$, it follows that $a^u = x$ and $a^v = y$. Now by law I of indices (theorem 2.22), we have

$$xy = a^u a^v = a^{u+v},$$

converting this equation to log form we get $\log_a(xy) = u + v = \log_a x + \log_a y$, proving I. Similarly by law II of indices, we have

$$\frac{x}{y} = \frac{a^u}{a^v} = a^{u-v},$$

and converting this equation to log form we get $\log_a\left(\frac{x}{y}\right) = u - v = \log_a x - \log_a y$, proving II. Finally by law III of indices, we have

$$a^u = x \implies (a^u)^c = x^c \implies a^{cu} = x^c,$$

and converting to log form we get $\log_a(x^c) = cu = c\log_a x$, proving III, as required. □

Example 4.11. Before calculators, these laws were essential to be able to evaluate logarithms using logarithm tables. Given that $\log_2 3 \approx 1.584$ and $\log_2 5 \approx 2.322$, we find $\log_2 5400$. Indeed, $\log_2 5400 = \log_2(2^3 \cdot 5^2 \cdot 3^3)$ by prime factor decomposition. By law I this becomes $\log_2(2^3) + \log_2(5^2) + \log_2(3^3)$, which then by law III becomes $3\log_2 2 + 2\log_2 5 + 3\log_2 3 = 3(1) + 2(2.322) + 3(1.584) = 12.396$.

Notation (Standard Bases)**.** Some logarithm bases are very common, so we give them their own special symbol.

(i) $\ln x$ denotes $\log_e x$ and is called the *natural logarithm*, where $e \approx 2.718$ is an important constant we will discuss later.

(ii) $\text{lb}\, x$ denotes $\log_2 x$, and is called the *binary logarithm*. This is common in computer science.

(iii) $\lg x$ denotes $\log_{10} x$, which is referred to the *common logarithm*.

(iv) $\log x$ without a base usually denotes the natural logarithm $\ln x$ in most mathematics and physics contexts, but in engineering it sometimes means $\log_{10}$. On scientific calculators, the log button usually stands for $\log_{10}$ also.

Examples 4.12. We solve some logarithmic equations.

(i) $\log x + \log 3 = \log(2x + 3)$.
Using the first law on the LHS, we get $\log(3x) = \log(2x + 3)$, and since log is injective, we can cancel it from both sides to get $3x = 2x + 3$, i.e., $x = 3$.

(ii) $2\log_7(x - 1) = \log_7 9$.
By the third law, this becomes $\log_7(x - 1)^2 = \log_7 9$, so $(x - 1)^2 = 9$, which implies that $x = 1 \pm 3$, i.e., $x = 4$ or $x = -2$.

But we need to be careful—recall from remark 3.4 that we cannot input negative numbers into a logarithm. In the original equation, the term $\log_7(x - 1)$ appears, so we can't input any $x \leqslant 1$ here. Thus the only valid solution is $x = 4$.

(iii) $\ln(2x+3) = 2\ln x - 5$.

There are two main approaches we can take with this equation. The first is to transform it into an indicial equation by taking the exponential $e^{(\cdot)}$ of both sides, this will give

$$e^{\ln(2x+3)} = e^{2\ln x - 5}.$$

The key thing to understand here is that $a^{(\cdot)}$ and $\log_a(\cdot)$ undo each other; e.g., $2^{\log_2 7} = 7$. This is the idea that motivates this approach. The equation becomes

$$2x+3 = e^{2\ln x-5} \implies 2x+3 = (e^{\ln x})^2 e^{-5} \implies 2x+3 = \frac{x^2}{e^5},$$

so we have the QE $x^2 - 2e^5 x - 3e^5 = 0$. Proceeding by completing the square,

$$(x-e^5)^2 - e^{10} - 3e^5 = 0 \implies x = e^5 \pm \sqrt{3e^5 + e^{10}}.$$

Now we need to be careful, remember that the equation we want to solve is $\ln(2x+3) = 2\ln x - 5$. Since $e^5 - \sqrt{3e^5 + e^{10}} \approx -1.49 < 0$, we cannot substitute this into the equation since $\ln x$ appears there. Thus the only valid solution is $x = e^5 + \sqrt{3e^5 + e^{10}}$.

The second approach is to use the laws of logarithms to get an equation of the form $\ln(\cdots) = \ln(\cdots)$, and cancel the logs as we did in (i) and (ii). Notice we can transform the coefficient 2 into a power on the RHS using the third law of logarithms:

$$\ln(2x+3) = 2\ln x - 5 \implies \ln(2x+3) = \ln\left(x^2\right) - 5$$

Now we can't do much with the -5, unless we write it as a logarithm, because then we can use the second law. Obviously $5 = \ln\left(e^5\right)$, so

$$\ln(2x+3) = \ln\left(x^2\right) - \ln\left(e^5\right) \implies \ln(2x+3) = \ln\left(\frac{x^2}{e^5}\right) \implies 2x+3 = \frac{x^2}{e^5},$$

and now we continue as in the first approach to get $x = e^5 + \sqrt{3e^5 + e^{10}}$.

Apart from allowing us to solve logarithmic equations, the laws of logarithms present us with another approach to solve indicial equations. Recall the equation $9^{3x} \cdot 3^{1+x} = 15^{2x+3}$ from examples 4.8. We can apply logarithms to both sides, and use the laws to bring the exponents down as terms in the equation. For instance, let's apply $\log_7$ to both sides:

$$\log_7(9^{3x} \cdot 3^{1+x}) = \log_7(15^{2x+3})$$

$$\begin{aligned}
\implies && \log_7(9^{3x}) + \log_7(3^{1+x}) &= (2x+3)\log_7 15 \\
\implies && 3x\log_7 9 + (1+x)\log_7 3 &= (2x+3)\log_7 15 \\
\implies && 3x\log_7 9 + \log_7 3 + x\log_7 3 &= 2x\log_7 15 + 3\log_7 15 \\
\implies && 3x\log_7 9 + x\log_7 3 - 2x\log_7 15 &= 3\log_7 15 - \log_7 3 \\
\implies && x(3\log_7 9 + \log_7 3 - 2\log_7 15) &= 3\log_7 15 - \log_7 3 \\
\implies && x &= \frac{3\log_7 15 - \log_7 3}{3\log_7 9 + \log_7 3 - 2\log_7 15}.
\end{aligned}$$

Just as at the start of this subsection, using a different method gave us a drastically different looking answer—in examples 4.8, we got $x = \log_{243/25} 1125$. But now we can use the laws of logs to try and massage the answer to look like the first one:

$$\begin{aligned}
\frac{3\log_7 15 - \log_7 3}{3\log_7 9 + \log_7 3 - 2\log_7 15} &= \frac{\log_7(15^3) - \log_7 3}{\log_7(9^3) + \log_7 3 - \log_7(15^2)} \\
&= \frac{\log_7(15^3/3)}{\log_7(9^3 \cdot 3/15^2)} \\
&= \frac{\log_7(1125)}{\log_7(\frac{243}{25})}.
\end{aligned}$$

Although this is not exactly the same answer as we got the first time round, it's a lot closer. The only thing we can't seem to get rid of is the $\log_7$, and if we think about it, the choice of 7 as a base was completely arbitrary. In fact, this example seems to suggest that we can just pick any base B and rewrite

$$\log_a x = \frac{\log_B x}{\log_B a}.$$

This is actually true! And it's called:

Theorem 4.13 (Change of Base)**.** *Let $a, x, B \in (0,\infty)$ where $a \neq 1 \neq B$. Then*

$$\log_a x = \frac{\log_B x}{\log_B a}.$$

Proof. Let $u = \log_a x$, so that $a^u = x$. Applying $\log_B$ to both sides, we get

$$\log_B(a^u) = \log_B x \implies u\log_B a = \log_B x \implies u = \frac{log_B x}{\log_B a},$$

which completes the proof. □

This theorem tells us that the logarithm in any base can be expressed in terms of logarithms of any other base. In fact in some textbooks, the only logarithm introduced is the natural logarithm $\ln x$, and then $\log_a x$ is defined to be short for $\ln x / \ln a$.

Now that we have developed these tools, we can easily resolve the issue we started this subsection with, namely to see why $\log_8 10 = \frac{1}{3}(\log_2 5 + 1)$. We can start by changing the base 8 to base 2:

$$\log_8 10 = \frac{\log_2 10}{\log_2 8} = \frac{\log_2(2\cdot 5)}{\log_2 8} = \frac{\log_2 5 + \log_2 2}{3} = \frac{\log_2 5 + 1}{3},$$

as required.

Example 4.14. We solve the equation $\log_x 9 + 1 = \log_3 x$. It's a bit strange to have an unknown in the base, perhaps we ought to invoke the change of base formula to get everything in terms of base 3:

$$\begin{aligned} \frac{\log_3 9}{\log_3 x} + 1 = \log_3 x &\implies \frac{2}{\log_3 x} + 1 = \log_3 x \\ &\implies 2 + \log_3 x = (\log_3 x)^2 \\ &\implies (\log_3 x)^2 - \log_3 x - 2 = 0 \end{aligned}$$

Here we have the quadratic $t^2 - t - 2$ where $t = \log_3 x$. This factorises to give

$$\begin{aligned} & (\log_3 x + 1)(\log_3 x - 2) = 0 \\ \implies \quad & \log_3 x = -1 \quad \text{or} \quad \log_3 x = 2 \\ \implies \quad & x = \tfrac{1}{3} \quad \text{or} \quad x = 9. \end{aligned}$$

Checking the original equation, we see that both answers are valid.

Exercise 4.15. 1. Express the following in terms of a, b and c; where $a = \log x$, $b = \log y$ and $c = \log z$.

a) $\log xyz$ b) $\log \dfrac{x}{zy}$ c) $4 \log y\sqrt{x}$

d) $\log 4x - \log 3y$ e) $\log(xy)^a$ f) $\ln x$

2. Solve the following equations.

a) $2^x = 32$ b) $7^x = 14$

c) $\log 5x = 1$

d) $3^{x^2-3x} = 81$

e) $2^{x+1} + 4 = 9(2^x)$

f) $216(2^{2n} + 3^{2n}) = 793(6^n)$

g) $12^{3x+1} \times 15^{5-2x} = 2^{2(2x+1)} \times 3^{3x} \times 5$

h) $3^{2x+1} = 3^{x+2} + \sqrt{1 - 6(3^x) + 3^{2(x+1)}}$

i) $\log 5 + \log 2x = 2$

j) $\dfrac{18\log_8 x - 8}{\log_8 x} = 9\log_8 x$

k) $4\log x = 2\log x - \log \dfrac{625}{4}$

l) $\operatorname{lb}(5 - x^2) = 2\operatorname{lb}(1 - x)$

m) $\log_x 27 - \log_x x = \dfrac{2}{\log_{27} 9}$

n) $\dfrac{\ln\left(35 - x^3\right)}{\ln(5 - x)} = 3$

o) $100x^{\log x - 2} + x^{2 - \log x} - 20 = 0$

p) $2 + \log\sqrt{1+x} + 3\log\sqrt{1-x} = \log\sqrt{1-x^2}$

3. Show that the unique solution of the equation $2^x 5^{2x} 7^{3x} = 3$ is given by the real number

$$x = \frac{\ln 3}{\ln 2 + 2\ln 5 + 3\ln 7}.$$

4. Solve the following systems of equations.

a) $\begin{cases} 2\log y + \log 2 = \log x \\ 5y = x + 2 \end{cases}$

b) $\begin{cases} \log_3 x \log_3 y = 6 \\ \log_3 xy = 5 \end{cases}$

c) $\begin{cases} 2\log_2 y + 2 = \log_2 x \\ x + y = 3 \end{cases}$

d) $\begin{cases} 2\log\left(y - 1\right) = \log x \\ 2x = 4 - y \end{cases}$

e) $\begin{cases} \log_2 x - 4 = \log_4 y \\ \log_2(x - 2y) = 5 \end{cases}$

f) $\begin{cases} \log_2 x + \log_4 y - 4 = 0 \\ 3^{x^2} - 9(3^{15y+2}) = 0 \end{cases}$

5. An amount of €1500 is deposited in a bank paying an annual interest rate of 5% compound interest per year. How many years must pass for this amount to exceed €2000?

6. A certain strain of E-coli bacteria doubles in number 30 minutes. If there are 100 E-coli bacteria that are allowed to grow under ideal conditions, how long will it take to reach 1 million bacteria?

7. The speed of the wind in a tornado, v (km/h), is related to the distance s (km) it travels before dying out by the equation $v = 93\log_6 s + 63$. If a tornado has wind speed of 95 m/s, how far does it travel?

8. Show that the solutions of the equation $2\log_2(x+15) - \log_2 x = 6$ are also the solutions of the equation $x^2 + 255 = 34x$ without actually finding the solutions.

9. Let $a, b > 0$. Show that if $\ln \frac{1}{2}(a+b) = \frac{1}{2}(\ln a + \ln b)$, then $a = b$.

10. Prove the following results WITHOUT USING THE CHANGE OF BASE THEOREM.

    a) $\log_{\sqrt{b}} x = 2\log_b x$
    b) $\log_{1/\sqrt{b}} \sqrt{x} = -\frac{1}{2}\log_b x$
    c) $\log_{b^4} x^2 = \log_b \sqrt{x}$

☕ 11. Prove that

$$\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_4 x} + \cdots + \frac{1}{\log_{100} x} = \frac{1}{\log_{100!} x},$$

where $100! = 100 \cdot 99 \cdot 98 \cdots 2 \cdot 1$.

(MATSEC May '17)

V. POLYNOMIALS

HERE we attempt to generalise the ideas from the section on quadratics to algebraic expressions containing larger integer powers of a single variable x. Can we hope to solve equations containing multiple powers of x?

THE BASICS AND POLYNOMIAL DIVISION

Definition 5.1 (Polynomial)**.** Let $n \in \mathbb{N} \cup \{0\}$, and let $a_0, a_1, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$. Any expression of the form

$$p = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is said to be a *polynomial in x*, where x is called an *indeterminate* and the a_i's are called its *coefficients*. The non-negative integer n is called the *degree* of the polynomial, denoted $\deg p$. The number a_n is called its *leading coefficient*.

Notation. The set of all polynomials with coefficients in $\mathbb{R}$ is denoted $\mathbb{R}[x]$. More generally, the set of polynomials with coefficients in some set F is denoted by $F[x]$, e.g., the set of polynomials with integer coefficients is $\mathbb{Z}[x]$.

Examples 5.2. We give some examples.

(i) The expression $12x^7 - 32x^3 + 5x^2 - 3x + 2$ is a polynomial of degree 7, since it is of the form $a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, where $a_7 = 12$, $a_3 = -32$, $a_2 = 5$, $a_1 = -3$, $a_0 = 2$ and $a_6 = a_5 = a_4 = 0$.

(ii) Any linear expression $ax + b$ is a polynomial of degree 1, since it is of the form $a_1x + a_0$.

(iii) Any quadratic $ax^2 + bx + c$ is a polynomial of degree 2, since it is of the form $a_2x^2 + a_1x + a_0$.

(iv) Any non-zero real number a_0 is a polynomial of degree 0.

Remark 5.3 (The zero polynomial). Notice that by definition 5.1, we have that $p = 0$ is technically not a polynomial, since we require that the lead coefficient (a_n) is always non-zero. In the definition, we do this so that the degree is defined meaningfully, otherwise we could say that something like $0x^2 + 3$ has degree 2.

Even though it does not agree with definition 5.1, we still consider the expression $p(x) = 0$ to be a polynomial, and we call it the *zero polynomial*.

Notation (Abstract polynomial)**.** We want to make the distinction between the notation p and $p(x)$ where p is a polynomial. When we write p alone, we are referring to the polynomial p in the abstract, in the sense that, p is the object with coefficients, p has a degree, and so on. On the other hand, $p(x)$ denotes a real number obtained when x is plugged into p. Therefore it makes no sense to speak of the degree of $p(x)$, or the coefficients of $p(x)$, since concretely, this is just a number.

Now we will abuse this convention slightly when it makes it convenient to write things down. For instance, when we write $ax^2 + bx + c$ as we did in the section on quadratics, then in most cases, we cared about the quadratic as an abstract entity, rather than its value at some particular x. We could write something abstract looking such as $a\square^2 + b\square + c$ to make the distinction clearer, but this looks a bit strange so we will stick to using x. But make sure that you are aware that a distinction is being made.

When we write things like $p + q$ or pq, we are referring to the polynomials obtained when they are treated as formal expressions and simplified algebraically. For instance, if $p = x^2 - 3$ and $q = 3x + 5$, then

$$p + q = (x^2 - 3) + (3x + 7) = x^2 + 3x + 4$$

and

$$pq = (x^2 - 3)(3x + 7) = 3x^3 + 7x^2 - 9x - 21.$$

Similarly, if we say that, e.g., "there exists a polynomial s such that $p = qs$", then this polynomial s is such that they are equal *as polynomials*, meaning that p and qs have the same coefficients and the same degree. Notice therefore that writing something like $p = q$ is different from writing $p(x) = q(x)$. By the former, we mean we have equality *as polynomials*, whereas the latter means we have equality for a particular value of x.

Example 5.4. If $p = x^2 - 3$ and $q = 3x + 7$, then $p \neq q$, but $p(x) = q(x)$ is possible (if x happens to be equal to 5, say).

Some textbooks say that two polynomials p and q are equal if $p(x) = q(x)$ for all possible inputs $x \in \mathbb{R}$. Is this equivalent to our definition, or is ours stronger? Obviously if $p = q$ by our definition, then $p(x) = q(x)$ for all inputs x, but if $p(x) = q(x)$ for all x, does it necessarily imply that the degree and coefficients of the polynomial are the same?

This turns out to be true (at least, for $F[x]$ when F is infinite), but we will prove it later in proposition 5.36. This fact also justifies the idea of "comparing coefficients" which we used to prove theorem 3.19 (Viète's formulæ for quadratics).

Definition 5.5 (Polynomial Factors)**.** Let $p, q \in \mathbb{R}[x]$ be polynomials. If there exists a polynomial $s \in \mathbb{R}[x]$ such that $p = qs$, then we say that q is a *factor* of p, or that q *divides* p. This relation is denoted by $q \mid p$, and the polynomial s is denoted by p/q.

Example 5.6. The polynomial $q = x-3$ is a factor of $p = 2x^2 - 7x + 3$, since p can be written as $p = (x-3)(2x-1)$. Similarly $s = x-2$ divides the polynomial $t = 6x^4 - 10x^3 - 7x^2 + 5x + 2$ because t can be written as $t = (x-2)(3x+1)(2x^2-1)$.

Remark 5.7. The polynomial p/q agrees with the usual notation for division of real numbers when it makes sense, i.e., if $p = qs$, then for any x, $s(x)$ will be equal to the value of the real number $p(x)/q(x)$, unless $q(x) = 0$. Notice however, that in the case that $q(x) = 0$, the value of $s(x)$ is still defined; so we can evaluate $(p/q)(x)$ $(\neq p(x)/q(x))$. This is one of the situations where the distinction between the notations p and $p(x)$ is important!

Example 5.8. In the last example, we had $p = (x-3)(2x-1)$ and $q = (x-3)$ and so $p = qs$ where $s = p/q = 2x-1$. Now $(p/q)(1) = s(1) = 2(1) - 1 = 1$, and also $p(1)/q(1) = ((1-3)(2(1)-1))/(1-3) = 1$. However, $(p/q)(3) = s(3) = 2(3) - 1 = 5$, but $p(3)/q(3)$ is not defined since $q(3) = 0$.

Remark 5.9. The degree of the *zero polynomial* $p = 0$ for all x is not defined. (In particular, it is *not* zero). The reason for this is that the results of the following theorem would not hold otherwise.

Theorem 5.10 (Degree Laws)**.** *Let $p, q \in \mathbb{R}[x]$ be non-zero polynomials. Then*

(i) $\deg(pq) = \deg p + \deg q$

(ii) *If $q \mid p$, then* $\deg(p/q) = \deg p - \deg q$

(iii) *If $n \in \mathbb{N}$, then* $\deg(p^n) = n \deg p$

Proof. For (i), simply observe that if we have $p = a_n x^n + \cdots + a_0$ and $q = b_m x^m + \cdots + b_0$, then

$$pq = (a_n x^n + \cdots + a_0)(b_m x^m + \cdots + b_0) = a_n b_m x^{n+m} + \cdots + a_0 b_0,$$

so $\deg(pq) = n + m = \deg p + \deg q$.

For (ii), if $q \mid p$, then by definition $p = qs$ for some $s \in \mathbb{R}[x]$. Therefore $\deg p = \deg(qs) = \deg q + \deg s = \deg q + \deg(p/q)$ by (i).

Finally for (iii), observe that by (i), $\deg(p^n) = \deg(p \cdots p) = \deg p + \cdots + \deg p = n \deg p$. □

Note. Observe the similarity between these degree laws and the laws of logarithms (theorem 4.10).

Example 5.11. These laws allow us to determine the degrees of polynomial products on inspection. For example, using law (i), we get

$$\deg((x+2)(3x-4)(x^2+1)) = 1+1+2 = 4,$$

and another example,

$$\deg((x-3)^2(5-2x)^3(x^3-9x+4)^2) = 2\cdot 1+3\cdot 1+2\cdot 3 = 11,$$

by (i) and (iii).

We also have the following property about the degree of sums of polynomials.

Proposition 5.12. *Let $p, q \in \mathbb{R}[x]$ be polynomials. Then*

$$\deg(p+q) \leqslant \max\{\deg p, \deg q\},$$

where $\max\{a, b\}$ *denotes the larger of the two numbers a and b.*

The proof is straightforward.

We can make sense of the notation p/q in the case that q does not divide p, by considering the corresponding algebraic expression in x, i.e., $p(x)/q(x)$.

Definition 5.13 (Rational Function)**.** A *rational function* is an algebraic expression of the form

$$\frac{p(x)}{q(x)}$$

where $p, q \in \mathbb{R}[x]$ are polynomials. Furthermore, if $\deg p < \deg q$, we say that p/q is *proper*. Otherwise if $\deg p \geqslant \deg q$ we say that p/q is *improper*.

Examples 5.14. The following are rational functions.

$$\frac{5x^2-6x}{x^3+5x-4} \qquad \frac{x^2-3x+2}{4x^2-5x+3} \qquad \frac{x^6}{x^5+x^2-4}$$

The first one is proper, the second two are improper.

In general, $(p/q)(x)$ equals $p(x)/q(x)$ for any x unless $q(x)=0$, in which case, it is either undefined, or can be assigned a value if p and q have common factors which cancel, so that we can evaluate it as in remark 5.7.

Example 5.15. Even though the numerator and denominator of

$$f(x) = \frac{x^2 - 1}{x^2 - 4x + 3}$$

are both zero when we plug in $x = 1$, we have $f(1) = -1$. Indeed, rather than treating it as a formal algebraic expression and substituting directly, we first notice that $(x-1)$ is a factor of both the numerator and denominator, so in fact we can simplify

$$f(x) = \frac{(x-1)(x+1)}{(x-1)(x+3)} = \frac{x+1}{x-3}.$$

Plugging in $x = 1$ into the simplified version of f allows us to assign the value -1 to f at $x = 1$.

Now, in primary school, fractions such as $\frac{7}{3}$ were described as improper, and we would instead write them as mixed numbers; so $\frac{7}{3}$ becomes $2\frac{1}{3}$ (i.e., $2 + \frac{1}{3}$). We could equivalently write what we're saying here as

$$7 = 3 \cdot 2 + 1.$$

The conversion from improper fractions to mixed numbers involved a process called *long division*. Here we introduce an analogue to long division for polynomials.

Theorem 5.16 (Euclidean Algorithm)**.** *Let $p, q \in \mathbb{R}[x]$ with $\deg p \geqslant \deg q$. Then we may write p as*

$$sq + r,$$

where $s, r \in \mathbb{R}[x]$ are polynomials such that:

- $\deg s = \deg p - \deg q$,
- $r = 0$ *or* $\deg r < \deg q$.

Proof. Suppose $p, q \in \mathbb{R}[x]$ are $p = a_n x^n + \cdots a_0$ and $q = b_m x^m + \cdots b_0$, where $\deg p = n \geqslant m = \deg q$. Define $s = \frac{a_n}{b_m} x^{n-m}$. The key is to observe that we can transform p by writing it as

$$p = sq + (p - sq),$$

where it's easy to see that $\deg(p - sq) < \deg p$, since the leading term $a_n x^n$ is eliminated. If $\deg(p - sq) < \deg q$, then we are done, if not, then we carry out the same transformation on $(p - sq)$, which gives

$$p = sq + s'q + ((p - sq) - s'q) = (s + s')q + (p - sq - s'q),$$

where we are denoting the "new" s by s'. Now $\deg(p - sq - s'q) < \deg(p - sq) < \deg p$. We keep doing this procedure until the degree of $p - sq - s'q - s''q - \cdots$ is less than $m = \deg q$ (or if it equals zero), at which point we are done. Clearly the degree of the obtained polynomial $s + s' + \cdots$ is $\deg(s) = n - m = \deg p - \deg q$. □

Corollary 5.17 (Euclidean Algorithm for Rational Functions)**.** *Let $p, q \in \mathbb{R}[x]$ with $\deg p \geqslant \deg q$. Then we may write the rational function p/q as $s + q/r$ where $s, r \in \mathbb{R}[x]$ are polynomials such that:*

- $\deg s = \deg p - \deg q$,
- *$r = 0$ or $\deg r < \deg q$ (so that r/q is proper).*

Example 5.18. The best way to understand the Euclidean algorithm (and its proof) is to work through an example. Let

$$p = 3x^4 - x + 1 \qquad \text{and} \qquad q = x^2 + 2x - 1.$$

In the proof, we defined s as the ratio of the leading term of p (i.e., $a_n x^n$) to that of q (i.e., $b_m x^m$), so for this example we have $s = 3x^4/x^2 = 3x^2$. Then we used the fact that $p = sq + (p - sq)$, and most importantly, since the term $a_n x^n$ appears both in p and in sq, the degree of $p - sq$ is less than that $n = \deg p$. Applying this reasoning to our example, we have

$$\begin{aligned} p &= 3x^2 \cdot q + \big(3x^4 - x + 1 - (x^2 + 2x - 1)(3x^2)\big) \\ &= 3x^2 \cdot q + \big(3x^4 - x + 1 - (3x^4 + 6x^3 - 3x^2)\big) \\ &= 3x^2 \cdot q + (-6x^3 + 3x^2 - x + 1), \end{aligned}$$

and as we can see, the degree of what's left over has decreased, however it's still not less than $2 = \deg q$. We therefore apply the same procedure again on the result, and the "new" s is now $-6x^3/x^2 = -6x$:

$$\begin{aligned} p &= 3x^2 \cdot q + \big(-6x \cdot q + \big(-6x^3 + 3x^2 - x + 1 - (-6x)(x^2 + 2x - 1)\big) \\ &= (3x^2 - 6x)q + \big(-6x^3 + 3x^2 - x + 1 - (-6x^3 - 12x^2 + 6x)\big) \\ &= (3x^2 - 6x)q + (15x^2 - 7x + 1). \end{aligned}$$

The degree of what's left over is still not less than 2, however one more iteration yields

$$\begin{aligned} p &= (3x^2 - 6x + 15)q + \big(15x^2 - 7x + 1 - 15(x^2 + 2x - 1)\big) \\ &= (3x^2 - 6x + 15)(x^2 + 2x - 1) + 16 - 37x, \end{aligned}$$

which is finally in the required form.

In terms of corollary 5.17, what this tells us is that we can write

$$\frac{p}{q} = \frac{sq + r}{q} = s + \frac{r}{q},$$

where now r/q is proper. In other words, we have

$$\frac{p}{q} = 3x^2 - 6x + 16 + \frac{16 - 37x}{x^2 + 2x - 1}.$$

Notation. In view of this method being the analogue to *long division* for polynomials, we adopt a similar way of denoting the procedure. Instead of proceeding as we have done in the example above, we instead write out the problem as a long division problem:

$$\begin{array}{r|ccccc}
\cline{2-6}
x^2+2x-1 & 3x^4 & & & -x & +1
\end{array}$$

Notice that we leave room for any terms in x^3 or x^2 to appear, organising the terms in non-overlapping "columns".

We start the algorithm just as before, by computing the 's' term, which is the result of dividing the left-most term below the division sign with the left-most term of the polynomial outside division sign. This is written above the division sign.

$$\begin{array}{r|ccccc}
 & & & 3x^2 & & \\
\cline{2-6}
x^2+2x-1 & 3x^4 & & & -x & +1
\end{array}$$

Next we multiply the term above by the polynomial outside the division sign to get the equivalent of sq. This is written underneath the polynomial below the division sign, and the signs are flipped, so that we have $-sq$.

$$\begin{array}{r|ccccc}
 & & & 3x^2 & & \\
\cline{2-6}
x^2+2x-1 & 3x^4 & & & -x & +1 \\
 & -3x^4 & -6x^3 & +3x^2 & & \\
\cline{2-4}
\end{array}$$

Next, we add to get $p - sq$. This will only differ from p in terms of degree 2 or higher, so we can just add those for now, and copy the $-x$ down for the next stage.

$$\begin{array}{rrrrrr}
 & & & 3x^2 & & \\ \cline{2-6}
x^2+2x-1\big) & 3x^4 & & & -x & +1 \\
 & -3x^4 & -6x^3 & +3x^2 & & \\ \cline{2-4}
 & & -6x^3 & +3x^2 & -x & \\
\end{array}$$

If $p - sq$ had degree less than 2, we would be done, but here the degree is 3, so we start the procedure again, dividing the leading coefficient of the remainder by the leading coefficient of the divisor, writing this "new s" in the appropriate column at the top.

$$\begin{array}{rrrrrr}
 & & & 3x^2 & -6x & \\ \cline{2-6}
x^2+2x-1\big) & 3x^4 & & & -x & +1 \\
 & -3x^4 & -6x^3 & +3x^2 & & \\ \cline{2-4}
 & & -6x^3 & +3x^2 & -x & \\
\end{array}$$

Multiplying this by q and flipping signs, we get the "new $p - sq$"

$$\begin{array}{rrrrrr}
 & & & 3x^2 & -6x & \\ \cline{2-6}
x^2+2x-1\big) & 3x^4 & & & -x & +1 \\
 & -3x^4 & -6x^3 & +3x^2 & & \\ \cline{2-4}
 & & -6x^3 & +3x^2 & -x & \\
 & & 6x^3 & +12x^2 & -6x & \\ \cline{3-5}
\end{array}$$

Adding the terms affected (those with degree 1 and higher) and copying the +1 down for the next stage, we get

$$\begin{array}{rrrrrr}
 & & & 3x^2 & -6x & \\ \cline{2-6}
x^2+2x-1\big) & 3x^4 & & & -x & +1 \\
 & -3x^4 & -6x^3 & +3x^2 & & \\ \cline{2-4}
 & & -6x^3 & +3x^2 & -x & \\
 & & 6x^3 & +12x^2 & -6x & \\ \cline{3-5}
 & & & 15x^2 & -7x & +1 \\
\end{array}$$

We check the degree again, and since it's not less less than 2, we repeat the procedure to finally arrive at

$$
\begin{array}{rrrrrr}
 & & & 3x^2 & -6x & +15 \\ \cline{2-6}
x^2+2x-1 \big) & 3x^4 & & & -x & +1 \\
 & -3x^4 & -6x^3 & +3x^2 & & \\ \cline{2-4}
 & & -6x^3 & +3x^2 & -x & \\
 & & 6x^3 & +12x^2 & -6x & \\ \cline{3-5}
 & & & 15x^2 & -7x & +1 \\
 & & & -15x^2 & -30x & +15 \\ \cline{4-6}
 & & & & -37x & +16
\end{array}
$$

Which corresponds to the fact that $3x^4 - x + 1 = (3x^2 - 6x + 15)(x^2 + 2x - 1) + 16 - 37x$, or in terms of rational functions,

$$\frac{3x^4 - x + 1}{x^2 + 2x - 1} = 3x^2 - 6x + 15 + \frac{-37x + 16}{x^2 + 2x + 1},$$

as we obtained previously.

Examples 5.19. Here are two more examples with this new notation. To divide $(x^3 - 8x^2 + 3x - 1)/(x^2 - 7x + 2)$, we do

$$
\begin{array}{rrrrr}
 & & & x & -1 \\ \cline{2-5}
x^2-7x+2 \big) & x^3 & -8x^2 & +3x & -1 \\
 & -x^3 & +7x^2 & -2x & \\ \cline{2-4}
 & & -x^2 & +x & -1 \\
 & & x^2 & -7x & +2 \\ \cline{3-5}
 & & & -6x & +1
\end{array}
$$

so

$$\frac{x^3 - 8x^2 + 3x - 1}{x^2 - 7x + 2} = x - 1 + \frac{1 - 6x}{x^2 - 7x + 2},$$

and for $(2x^2 - 2x + 2)/(3x^2 + 3x + 1)$, we do

$$
\begin{array}{rrrr}
 & & & \frac{2}{3} \\ \cline{2-4}
3x^2+3x+1 \big) & 2x^2 & -2x & +2 \\
 & -2x^2 & -2x & -\frac{2}{3} \\ \cline{2-4}
 & & -4x & +\frac{4}{3}
\end{array}
$$

so

$$\frac{2x^2 - 2x + 2}{3x^2 + 3x + 1} = \frac{2}{3} - \frac{12x - 4}{3(3x^2 + 3x + 1)}.$$

Exercise 5.20. 1. State whether the following rational functions are proper or improper.

a) $\dfrac{4x^2+7x-3}{2x+1}$ b) $\dfrac{4t+1}{3t-1}$

c) $\dfrac{4x^2-28}{3x}$ d) $\dfrac{x^3+x^2+x+1}{x^2+x+1}$

e) $\dfrac{4x^4+3x^3+2x^2+x}{x+1}$ f) $\dfrac{(x-1)(x-2)(x-3)}{(x+1)(x+2)}$

g) $\dfrac{4x^3+x}{16x^6-x^2}$ h) $\dfrac{1}{x+1}(x^3-27)$

i) $\dfrac{\pi x+2\pi+\pi^2}{2(x+\pi)}$ j) $\dfrac{x^4-x^2-2x-1}{x^2+x+1}$

2. Convert the improper fractions from the exercise above to proper fractions.

Now we will conclude with some more theory which will be useful in future sections.

Definition 5.21 (Monic)**.** A polynomial p is said to be *monic* if its leading coefficient is 1, i.e., if $\deg p = n$, then the coefficient of x^n in p is 1.

Recall that the *highest common factor* of two integers a and b is the largest positive integer k such that $k \mid a$ and $k \mid b$ (i.e., k divides both a and b). For example, the highest common factor of 15 and 25 is 5, and that of 27 and 36 is 9.

Definition 5.22 (Polynomial hcf)**.** Let p, q be two polynomials (not both zero). Then the highest common factor of p and q, denoted $\mathrm{hcf}(p,q)$, is the monic polynomial s of largest degree such that $s \mid p$ and $s \mid q$.

Example 5.23. If $p = x^4+2x^3+x+2$ and $q = 3x^4-4x^3-43x^2-56x-20$, we have $\mathrm{hcf}(p,q) = x^2+3x+2$, since in factorised form, these polynomials are $p = (x+1)(x+2)(x^2-x+1)$, $q = (x+1)(x+2)(x-5)(3x+2)$ and $x^2+3x+2 = (x+1)(x+2)$. Notice we require the additional constraint that the hcf is monic so that it is unique, otherwise any constant multiple of the hcf still divides both p and q.

An important fact about the hcf is the following.

Proposition 5.24. *Let $p, q \in \mathbb{R}[x]$ be two polynomials (not both zero). Then the highest common factor* $\mathrm{hcf}(p,q)$ *exists and is unique.*

We will not prove this fact because it requires some advanced algebra (namely the theory of Euclidean rings). But it plays a key role in the proof of the following result.

Theorem 5.25 (Bézout's lemma)**.** *Let $p, q \in \mathbb{R}[x]$ be two polynomials. Then there exist $s, t \in \mathbb{R}[x]$ such that*

$$sp + tq = \operatorname{hcf}(p, q),$$

where $\deg(s) < \deg(q)$ *and* $\deg(t) < \deg(p)$.

Notice that an an analogous result is true for integers: for any $a, b \in \mathbb{Z}$, we can find $s, t \in \mathbb{Z}$ such that $sa + tb = \operatorname{hcf}(a, b)$. For instance, if $a = 3$ and $b = 5$, then

$$2 \cdot 3 + (-1) \cdot 5 = 1 = \operatorname{hcf}(3, 5).$$

We will not prove this result either (not for integers, nor for polynomials), but it is essentially a consequence of theorem 5.16 plus some advanced algebra.

THE REMAINDER AND FACTOR THEOREMS

With numbers, the term "remainder" refers to the quantity left over when performing division. For example, if we divide 17 by 3, then we get

$$\frac{17}{3} = 5 + \frac{2}{3},$$

so the remainder is 2. Notice that the remainder is always less than the divisor. For polynomials, we have the analogous notion, where we perform long division to get

$$\frac{p}{q} = s + \frac{r}{q},$$

and $\deg(r) < \deg(q)$ (or $r = 0$). Sometimes it is more useful to look at this relation in the form

$$p = qs + r.$$

The corresponding equality for the numeric example would be $17 = 3 \cdot 5 + 2$ (this is why we stated theorem 5.16 the way we did). But unlike numbers, polynomials have a variable which we can substitute for. This allows us to prove the following.

Theorem 5.26 (Remainder theorem)**.** *Let $p \in \mathbb{R}[x]$ be a polynomial. Then the remainder upon division by $(x - \alpha)$ is $p(\alpha)$.*

Proof. First of all notice that when we divide p by $(x - \alpha)$, by theorem 5.16, the remainder has $r(x) < \deg(x - \alpha) = 1$ or it equals 0, i.e., the remainder is a constant. After division, we can write $p = s \cdot (x - \alpha) + r$. In particular, this is true when

$x = \alpha$, which gives us that

$$p(\alpha) = s(\alpha)(0) + r,$$

i.e., that $r = p(\alpha)$. □

This allows us to determine the remainder of a division without having to actually perform the division! (Albeit limited to the case where our divisor is linear.)

Example 5.27. Say we divide $p(x) = x^4 + 3x^3 + 5x^2 + 7x + 9$ by $x + 1$. Then the remainder should be equal $p(-1)$ (since $(x+1) = (x-(-1))$), i.e.,

$$r = p(-1) = (-1)^4 + 3(-1)^3 + 5(-1)^2 + 7(-1) + 9 = 5.$$

Indeed, if we do the division, we get

$$\begin{array}{r|l} & x^3 + 2x^2 + 3x + 4 \\ \hline x+1 & x^4 + 3x^3 + 5x^2 + 7x + 9 \\ & -x^4 - x^3 \\ \hline & 2x^3 + 5x^2 \\ & -2x^3 - 2x^2 \\ \hline & 3x^2 + 7x \\ & -3x^2 - 3x \\ \hline & 4x + 9 \\ & -4x - 4 \\ \hline & 5 \end{array}$$

which gives

$$\frac{x^4 + 3x^3 + 5x^2 + 7x + 9}{x+1} = x^2 + 2x + 3x + 4 + \frac{5}{x+1},$$

as expected.

Remark 5.28. Notice that if we are dividing by a factor of the form $(ax+b)$, for the purposes of the remainder theorem, this is equivalent to dividing by $(x - (-{}^b/_a))$. Indeed, if

$$p = s \cdot (ax + b) + r,$$

putting $x = -{}^b/_a$ gives $p(-{}^b/_a) = s0 + r$, i.e., $r = p(-{}^b/_a)$.

Why is this theorem particularly useful? Why should we care about being able to find the remainder if we still have to do long division for the quotient? Well, if it happens that the remainder is zero, then we get

$$p = q \cdot (x - \alpha) + r = (x - \alpha) \cdot q,$$

i.e., we will have determined a linear factor of p. In other words, we have

Theorem 5.29 (Factor theorem)**.** *Let $p \in \mathbb{R}[x]$ be a polynomial. Then*

$$p(\alpha) = 0 \quad \text{if and only if} \quad (x - \alpha) \mid p.$$

Proof. If $p(\alpha) = 0$, then by the remainder theorem, we get that $p = (x - \alpha) \cdot s$.

Conversely, if $(x - \alpha) \mid p$, then $p = (x - \alpha) \cdot s$ and so $p(\alpha) = 0$. □

This allows us to prove various fundamental facts about polynomials. But first we will need to generalise it slightly. If α is a root of p, then by the factor theorem, $(x-\alpha) \mid p$, and so $p/(x-\alpha)$ is a polynomial. Now if α is a root of this polynomial, then applying the factor theorem again gives us that $(x - \alpha) \mid (p/(x-\alpha))$, or, $(x-\alpha)^2 \mid p$. But this in turn implies that $p/(x-\alpha)^2$ is a polynomial, and we can ask again whether α is a root of this new polynomial. If this is the case, we will similarly get that $(x - \alpha)^3 \mid p$, and so on. This idea gives rise to the following definition.

Definition 5.30 (Multiplicity)**.** Let $f \in \mathbb{R}[x]$ be a polynomial, and let $\alpha \in \mathbb{R}$ be a real root of f, so that $(x-\alpha) \mid f$ by the factor theorem and so $f/(x-\alpha)$ is a polynomial. Then α is said to be *repeated* if it is also a root of $f/(x-\alpha)$, or equivalently, if $(x-\alpha)^2 \mid f$.

More generally, we say that α is a *root with multiplicity n if* n is the largest integer such that $(x - \alpha)^n \mid f$.

Notice this definition is compatible with the notion of repeated root for quadratics introduced in theorem 3.12. If we say that a polynomial has roots $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, we do not exclude the possibility that some of the α_i are equal to each other, and interpret n occurrences of the same α_i as its multiplicity.

Examples 5.31. The polynomial $x^3 - 1$ has one root, namely $x = 1$ with multiplicity 1, since it factorises as $(x-1)(x^2+x+1)$ and the second factor has discriminant -3. The polynomial $x^5 + 3x^4 + 4x^3 + 4x^2 + 3x + 1$ has only one root, but it has multiplicity 3. Indeed, its factorisation is $(x-1)^3(x^2+1)$.

Now we can state a generalisation of theorem 3.14.

Theorem 5.32. *Let $p \in \mathbb{R}[x]$ be a polynomial with s real roots $\alpha_1, \ldots, \alpha_s \in \mathbb{R}$ (not necessarily distinct). Then*

$$p = (x - \alpha_1) \cdots (x - \alpha_s) \cdot q,$$

where q is either a polynomial with $\deg(q) = \deg(p) - s$ *or the zero polynomial.*

Proof. If p is the zero polynomial, then the result is obvious, we can take any $\alpha_i \in \mathbb{R}$ we want and then set q to be the zero polynomial. So suppose $p \neq 0$. By the factor theorem, $p(\alpha_1) = 0$, so we may express p as $p = (x - \alpha_1) \cdot t$. Now if $t(\alpha_1) = 0$ (i.e., α_1 is a repeated root), then we can apply the factor theorem to t and get that $p = (x - \alpha_1)^2 \cdot u$. Repeating the process until α_1 is no longer a root, we will obtain $p = (x - \alpha_1)^{m_1} \cdot v$, where $v(\alpha_1) \neq 0$, and m_1 denotes the multiplicity of α_1. (This is guaranteed to happen eventually by theorem 5.16, since dividing p by $(x - \alpha_1)$ repeatedly decreases its degree, and a degree 0 polynomial has no roots.)

Now if there are roots left (i.e., if they weren't all equal to α_1), then there is a root α_2 (let's keep the subscripts simple) so that $\alpha_2 \neq \alpha_1$ and

$$0 = p(\alpha_2) = (\alpha_1 - \alpha_2)^{m_1}\, v(\alpha_2).$$

Since $(\alpha_1 - \alpha_2)^{m_1} \neq 0$, then we must have $v(\alpha_2) = 0$, and so $(x - \alpha_2) \mid v$, and we can write $p = (x - \alpha_1)^{m_1}(x - \alpha_2) \cdot w$. Continuing similarly, accounting for repeated roots, we get that $p = (x - \alpha_1) \cdots (x - \alpha_s) \cdot q$ and $q \neq 0$.

Finally by theorem 5.10(i), we see that $\deg(p) = s + \deg(q)$. □

As an immediate consequence, we have an upper-bound to the number of roots of a polynomial.

Theorem 5.33. *Let $p \in \mathbb{R}[x]$ be a non-zero polynomial of degree n. Then p can have at most n roots (including multiplicity).*

Proof. Suppose p has s roots $\alpha_1, \ldots, \alpha_s$. Then $p = (x - \alpha_1) \cdots (x - \alpha_s) \cdot q$ with $q \neq 0$, and so

$$\begin{aligned}
\deg(p) &= \deg((x - \alpha_1) \cdots (x - \alpha_s) \cdot q) \\
&= \deg((x - \alpha_1) \cdots (x - \alpha_s)) + \deg(q) \\
&\geqslant \deg((x - \alpha_1) \cdots (x - \alpha_s)) = s,
\end{aligned}$$

so number of roots is at most the degree, as required. □

In general, the roots of polynomial equations of degree higher than quadratic are not as simple to understand. But what can these theorems tell us about the next type of polynomial, namely, those of degree 3? These are called *cubics.* Well, we know that the cubic can have at most 3 roots by theorem 5.33. In particular, it can have zero, one or three real roots (including multiplicity), since if it has one root, it will be equal to

$$(x - \alpha_1) \cdot q$$

where q has degree two, but if it has two roots, it will equal

$$(x-\alpha_1)(x-\alpha_2)\cdot q$$

and q will be linear, i.e., $q = a(x-\alpha_3)$, which necessarily introduces a third root. So a cubic cannot have precisely two roots. Moreover, we will later see that, unlike a quadratic, a cubic ax^3+bx^2+cx+d cannot have zero roots. The intuitive reason for this is that when x is very large in size (think of $x = 10\,000$), then x^3 is much larger than the remaining terms, and the value of the polynomial at x is approximately ax^3. Thus assuming $a > 0$, if x is large and positive, the value of the polynomial at x is large and positive, and similarly if x is large and negative, then the value of the polynomial at x is large and negative. (If $a < 0$, then things are the other way around). Either way, we have that at some point, the polynomial is positive, and at another point, the polynomial is negative. Because polynomials behave "nicely" (they are continuous, as we will discuss later), we will see that they must take on all values in between these extremes, and in particular, the cubic must take on the value zero for some x between these two "large" inputs. Thus, a cubic has at least one root.

In conclusion, a cubic always has either 1 or 3 real roots.

But this is not very helpful, since the roots are rarely easy to obtain unless the polynomial is hand-picked. (i.e., if I give you the cubic $(x-1)(x-2)(x-3)$, then clearly its roots are simple, but if I randomly pick coefficients for x^3, x^2, x and 1, even if they are all integers, then the roots can be complicated to express.) Take the innocent looking x^3+4x-1. Then this cubic only has one real root, and it is given by

$$4\sqrt[3]{\frac{2}{3(-9+\sqrt{849})}}-4\sqrt[3]{\frac{2}{3(9+\sqrt{849})}},$$

and the reasoning to obtain such an expression can sometimes involve nontrivial algebraic difficulties. This was not a specially picked polynomial whose root is particularly ugly, these were literally the first three random numbers I picked as coefficients. This is what "normal" cubic roots look like. The picture gets a lot worse for polynomials of higher degree. It turns out that if a fourth degree polynomial has roots, then they will be expressible (usually as some horrible humongous expression) involving square ($\sqrt{\ }$), cube ($\sqrt[3]{\ }$) and fourth ($\sqrt[4]{\ }$) roots. But for degree 5 and higher, most roots cannot be written down using any combination of nth roots, for n as large as you like, and we can only speak of their existence and approximate them numerically as decimals. The very profound and elegant theory behind these facts is called Galois theory, which was developed by the French mathematician Évariste Galois in the 19th century. He died at the age of 20 in a pistol duel (allegedly over a woman), and he was so convinced of his impending death that he stayed up all night writing letters to his Republican friends and composing what would become his mathematical testament, the famous letter to Auguste Chevalier outlining his ideas, and three attached manuscripts.

Therefore, when it comes to polynomials of degree higher than 2, we will restrict our interests only to roots which are rational. In particular, we have the following condition for the existence of rational roots which we will find useful.

Theorem 5.34 (Rational roots theorem)**.** *Let $p \in \mathbb{Z}[x]$ be a polynomial*

$$p = a_n x^n + \cdots + a_0$$

with integer coefficients. Then if $x = {}^c/_d$ is a root, $c \mid a_0$ and $d \mid a_n$.

Proof. We can assume that c and d share no common factors (i.e., the fraction is in its lowest form). If $p({}^c/_d) = 0$, then

$$\begin{aligned} a_n\left(\frac{c}{d}\right)^n + a_{n-1}\left(\frac{c}{d}\right)^{n-1} + \cdots + a_1\left(\frac{c}{d}\right) + a_0 = 0 \qquad (\times d^n) \\ \implies c(a_n c^{n-1} + a_{n-1}c^{n-2}d + \cdots + a_1 d^{n-1}) = -a_0 d^n, \end{aligned}$$

since both sides are integers, we get that c divides $-a_0 d^n$. Moreover, since c and d share no factors, it follows that $c \mid a_0$. On the other hand, we can rearrange the last equation to get

$$d(a_{n-1}c^{n-1} + \cdots + a_1 c d^{n-2} + a_0 d^{n-1}) = -a_n c^n,$$

and by similar reasoning, $d \mid a_n$. □

Example 5.35. The polynomial $2x^3 + 7x^2 + 16x + 15$ has $x = -{}^3/_2$ as a root. Indeed, $3 \mid 15$ (the constant coefficient), and $2 \mid 2$ (the leading coefficient). Notice that this is a *necessary* but *not sufficient* condition for the existence of a rational root. For instance, for the same polynomial, we have $5 \mid 15$ and $1 \mid 2$, but $5 = {}^5/_1$ is not a root.

Therefore in general, given an nth degree polynomial

$$a_n x^n + \cdots + a_0,$$

with integer coefficients, we have a method to obtain all possible rational roots/factors. If $c_0, \ldots, c_k$ are all the positive divisors of a_0 and $d_0, \ldots, d_\ell$ are all the positive divisors of a_n, then we try substituting

$$x = \pm\frac{c_i}{d_j}$$

for all $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \ell$. By theorem 5.34, this will exhaust all possible rational roots.

Before we give an example of the method, we will give this useful proposition.

Proposition 5.36 (Comparing coefficients)**.** *Let $f, g \in \mathbb{R}[x]$ be two non-zero polynomials, and let n be the larger of $\deg(f)$ and $\deg(g)$. If $f(x) = g(x)$ for $n+1$ different values of x, then $f(x) = g(x)$ for all values of x and moreover, f and g have the same degree and coefficients (i.e., $f = g$).*

Proof. If $f(x) = g(x)$ for $n+1$ different values of x, then $f - g$ has $n+1$ roots. But $\deg(f-g) \leqslant n$, so by theorem 5.33, $f-g$ is the zero polynomial, i.e., $f(x) = g(x)$ for all values of x.

Now for the coefficients, notice that the constant term is given by $f(0) = g(0)$, so they have the same constant term, say a_0. Define a new polynomial f_1 by $f_1(x) = (f(x) - a_0)/x$ and similarly $g_1(x) = (g(x) - a_0)/x$. These are polynomials since we can factor x out of both numerators. Now we have $f_1(0) = g_1(0)$ since these depend only on f and g which are equal for all x. But $f_1(0)$ is the coefficient of x in f and $g_1(0)$ is the coefficient of x in g, so their x-coefficients are equal, say to a_1. Now define $f_2(x) = (f_1(x) - a_1)/x$ and $g_2(x) = (g_1(x) - a_1)/x$. Continuing this way up to f_n and g_n, we get that all coefficients are equal. In particular, they have the same largest non-zero coefficient, which implies that their degrees are also equal. □

Thus from the proposition, we know that, for example, if $x^2 + 1 = ax^2 + bx + c$ for 3 different values of x, then we must have $a = 1$, $b = 0$ and $c = 1$. Now let us give an example of our general method for finding rational roots.

Example 5.37. Say we want to factorise the polynomial

$$p = 4x^4 - 4x^3 + 13x^2 - 12x + 3.$$

The divisors of 3 are $1, 3$ and the divisors of 4 are $1, 2, 4$. Thus we need to try

$$x = \pm\tfrac{1}{1}, \pm\tfrac{1}{2}, \pm\tfrac{1}{4}, \pm\tfrac{3}{1}, \pm\tfrac{3}{2}, \pm\tfrac{3}{4}.$$

If we do this, we get that only $p(1/2) = 0$, and so $p = (2x-1) \cdot q$ for some cubic q (notice this is equivalent to the factor $(x - 1/2)$). Now we could determine q using long division since $q = p/(2x-1)$. But a shorter way is possible. Since q is cubic, we have

$$p = (2x-1)(ax^3 + bx^2 + cx + d).$$

Moreover, it's easy to see that when we expand this, the constant term is $-d$ and the leading coefficient is $2a$. By proposition 5.36, these must be equal to the coefficients 3 and 4 of p (since we want equality for all $x \in \mathbb{R}$, let alone $n+1$ values!). Thus we get

$$p = (2x-1)(2x^3 + bx^2 + cx - 3).$$

Now to determine b and c, we can think about what the other coefficients are when we expand the brackets. The coefficient of x^3 is going to be $2b-2$, and this should equal -4, so $b = -1$. Similarly the coefficient of x will be $-c-6$, and this should equal -12, so $c = 6$. Therefore

$$p = (2x-1)(2x^3 - x^2 + 6x - 3).$$

The remaining coefficient, that of x^2, can be used to check that we've done things correctly. Indeed, this should equal $-b+2c = 1+12 = 13$, which agrees with p.

Now we need to check if we can factorise $2x^3 - x^2 + 6x - 3$ further. Clearly if this has a rational root, then it would also be a root of p, so we would have found it in the initial stage where we computed all possible rational roots. So this definitely can't have any "new" factors, but it might have another factor of $(2x-1)$, which corresponds to $x = {}^1\!/_2$ having multiplicity 2. Indeed, if we plug in $x = {}^1\!/_2$, we get that it equals zero again. Thus

$$2x^3 - x^2 + 6x - 3 = (2x-1)(ax^2 + bx + c)$$

for some quadratic $ax^2 + bx + c$. Again we can reason about the first and last coefficients upon expansion to get that

$$2x^3 - x^2 + 6x - 3 = (2x-1)(x^2 + bx + 3),$$

and for b, consider the coefficient of x^2. Upon expansion of the RHS, this is $2b-1$, and this should equal -1, so we get that $b = 0$. It follows that

$$p = (2x-1)^2(x^2+3).$$

This quadratic factor is not zero when $x = {}^1\!/_2$, so this root has multiplicity 2. Moreover, it has no real roots at all since its discriminant is -12. Therefore the solutions of the equation $p(x) = 0$ are $x = {}^1\!/_2$ (twice).

Example 5.38. Say we want to solve $x^4 + 12x^2 + 2 = 6x^3 + 9x$. First we rearrange it into a question of finding zeros, namely, the zeros of

$$x^4 - 6x^3 + 12x^2 - 9x + 2.$$

The only possible rational roots are $x = \pm 1, \pm 2$ by the rational roots theorem (5.34). Indeed, if we try these, we get that $(x-1)$ and $(x-2)$ are both factors. Hence

$$x^4 - 6x^3 + 12x^2 - 9x + 2 = (x-1)(x-2)(x^2 + bx + 1)$$

where the constant and leading term of the remaining factor were determined by thinking about expansion, as in the last example. Finally for b, notice that the coefficient of x^3 will be $b-1-2$ upon expansion, which should equal -6, so $b = -3$. Thus the equation becomes

$$(x-1)(x-2)(x^2 - 3x + 1) = 0$$
$$\implies x = 1 \quad \text{or} \quad x = 2 \quad \text{or} \quad x^2 - 3x + 1 = 0.$$

Solving this remaining quadratic by completing the square, we get that the solutions of the equation are $x = 1, 2, {}^1\!/_2(3 \pm \sqrt{5})$.

Exercise 5.39. 1. The polynomial $ax^3 - x^2 + bx - 6$ leaves a remainder of 54 when divided by $(x-4)$, whereas $(x-3)$ is a factor of the expression. Determine the values of a and b, and hence; by comparing coefficients or otherwise, express the given expression as a product of three linear factors.

2. Solve the following equations by factorising.

   a) $x^3 - 4x^2 + 10 = 3x - 8$
   b) $3x^3 - 10x^2 - 71x = 42$
   c) $0 = 6 + 7x - 9x^2 + 2x^3$
   d) $x^3 - 2x^2 - 36x + 7 = 0$
   e) $x^2(x^2 + 6x + 7) = 6x + 8$
   f) $x^4 + 16 = 8x^2$
   g) $24x^3 + 26x^2 + 9x + 1 = 0$
   h) $x^3 - 6x^2 + 10x = 3$
   i) $x^3 + 3a = ax^2 + 3x$

3. Let $p \in \mathbb{R}[x]$ be defined by $p = 3x^4 - 26x^3 + 39x^2 + 4x - 4$.

   a) Determine all the rational roots of p.
   b) Find the remaining roots of p *without* performing long division.
   c) What is the remainder when dividing p by $(x-3)$?

4. Solve the following cubic equations.

   a) $2x^3 - 15x^2 + 22x + 15 = 0$
   b) $3x^3 - 5x^2 + x + 1 = 0$
   c) $2x^3 + 7 = x(11x + 9)$
   d) $2x^3 - 11x^2 + 19x - 7 = 0$

5. Suppose $x = {}^3\!/_5$ is a root of the polynomial $a_n x^n + \cdots + a_0$ where $a_0, \ldots, a_n \in \mathbb{Z}$. Prove, *without using the rational roots theorem*, that $3 \mid a_0$ and that $5 \mid a_n$.

6. In this question, we will extend the theory of Viète's formulæ (theorem 3.19) to cubics. Suppose a cubic

$$p = ax^3 + bx^2 + cx + d$$

has roots $\alpha, \beta, \gamma \in \mathbb{R}$.

   a) Show that $p = a(x-\alpha)(x-\beta)(x-\gamma)$.

   b) Let $\Sigma = \alpha + \beta + \gamma$, $\Pi = \alpha\beta\gamma$ and $\Xi = \alpha\beta + \beta\gamma + \gamma\alpha$. Show that

$$p = a(x^3 - \Sigma x^2 + \Xi x - \Pi).$$

   c) Show that

$$\alpha^2 + \beta^2 + \gamma^2 = \Sigma^2 - 2\Xi$$

   and that

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\Xi^2 - 2\Sigma\Pi}{\Pi^2}.$$

   Hence, given that the cubic $3x^3 - 5x + 1$ has roots α, β, γ, determine a cubic with roots $\alpha/\beta\gamma$, $\beta/\alpha\gamma$ and $\gamma/\alpha\beta$.

7. a) By considering the polynomial $x^2 - 2$, use the rational roots theorem to deduce that it has no rational roots. Conclude that $\sqrt{2}$ is irrational.

   b) Construct a polynomial which has $\sqrt{2+\sqrt{3}}$ as a root. Use the rational roots theorem similarly to part (a) to conclude that it is irrational.

   c) Construct a quadratic polynomial which has $\sqrt{4+2\sqrt{3}} - \sqrt{3}$ as a root. Deduce that it is rational, and write it in the form a/b with $a, b \in \mathbb{Z}$.

PARTIAL FRACTIONS

It is simple to verify that, for example,

$$\frac{2}{x+1} - \frac{1}{x-2} = \frac{x-5}{(x+1)(x-2)}.$$

But can we somehow reverse this process? In other words, given

$$\frac{x-5}{(x+1)(x-2)},$$

can we decompose it as a sum of two rational functions over each linear denominator? First of all, observe that this is a proper rational function, and that its decomposition is a sum of proper rational functions. Indeed, if p_1, q_1, p_2, q_2 are polynomials with $\deg(p_1) < \deg(q_1)$ and $\deg(p_2) < \deg(q_2)$, then

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$$

is necessarily proper, since

$$\begin{aligned}
\deg(p_1 q_2 + p_2 q_1) &\leqslant \max\{\deg(p_1 q_2), \deg(p_2 q_1)\} \\
&= \max\{\deg(p_1) + \deg(q_2), \deg(p_2) + \deg(q_1)\} \\
&< \max\{\deg(q_1) + \deg(q_2), \deg(q_2) + \deg(q_1)\} \\
&= \deg(q_1) + \deg(q_2) = \deg(q_1 q_2),
\end{aligned}$$

where $\max\{a, b\}$ denotes the larger number of a and b.

But why would it be desirable to reverse this process? In later chapters, we are going to be performing operations on rational functions which are *additive*. An operation T is additive if

$$T(f+g) = T(f) + T(g).$$

Therefore, if we can decompose a rational function p/q into a sum of simpler rational functions $p_1/q_1 + \cdots + p_n/q_n$, it might turn out to be simpler to work out $T(p_1/q_1)$, ..., $T(p_n/q_n)$ separately and add them up, rather than to compute $T(p/q)$ directly. These will be the same if T is additive. (For those who are curious, additive operations include limits, summation, differentiation and integration). For now we will not worry about the applications, but suffice it to say that what we are doing will actually prove useful!

Let's start by considering rational functions of the form

$$\frac{f}{p_1 p_2}$$

where p_1 and p_2 do not have any factors in common (i.e., $\mathrm{hcf}(p_1, p_2) = 1$), and $\deg f < \deg(p_1 p_2)$, so that the function is proper.

Proposition 5.40. *Let* $f, p_1, p_2 \in \mathbb{R}[x]$ *with* $\operatorname{hcf}(p_1, p_2) = 1$*), and* $\deg f < \deg(p_1 p_2)$. *Then there exist polynomials* $f_1, f_2 \in \mathbb{R}[x]$ *with* $\deg f_1 < \deg p_1$ *and* $\deg f_2 < \deg p_2$ *such that*

$$f = f_1 p_2 + f_2 p_1.$$

Proof. By theorem 5.25 (Bézout's lemma), we can find polynomials s and t such that $sp_1 + tp_2 = \operatorname{hcf}(p_1, p_2) = 1$. So

$$f = f \cdot 1 = f(sp_1 + tp_2) = ftp_2 + fsp_1.$$

Now write $ft = up_1 + f_1$ with $\deg f_1 < \deg p_1$ by dividing ft by p_1 (theorem 5.16). Define $f_2 = fs + up_2$. Then

$$f = ftp_2 + fsp_1 = (up + f_1)p_2 + (f_2 - up_2)p_1 = f_1 p_2 + f_2 p_1.$$

We already have that $\deg f_1 < \deg p_1$, we just need to show that $\deg f_1 < \deg f_2$. By the above, we have that $f_2 = (f - f_2 p_1)/p_1$, so

$$\begin{aligned}
\deg f_2 &= \deg(f - f_1 p_2) - \deg p_1 \\
&\leqslant \max\{\deg f, \deg(f_1 p_2)\} - \deg p_1 \\
&< \max\{\deg(p_1 p_2), \deg(p_1 p_2)\} - \deg p_1 \\
&= \deg(p_1 p_2) - \deg p_1 = \deg p_2,
\end{aligned}$$

which completes the proof. □

Thus we have established the *existence* of a way to write

$$\frac{f}{p_1 p_2} = \frac{f_1 p_2 + f_2 p_1}{p_1 p_2} = \frac{f_1}{p_1} + \frac{f_2}{p_2},$$

where the two rational functions on the right are both proper. Let's take the example we started with,

$$\frac{x-5}{(x+1)(x-2)}.$$

By proposition 5.40, we can write

$$x - 5 = f_1(x-2) + f_2(x+1), \tag{$*$}$$

where $\deg f_1 < \deg(x+1)$ and $\deg f_2 < \deg(x-2)$, i.e., they are both degree zero polynomials (or zero). Assuming $f_1(x) = A$ and $f_2(x) = B$ for all x, we have

$$x - 5 = A(x-2) + B(x+1).$$