COORDINATE GEOMETRY

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ABSTRACT. In this set of notes, we introduce the basics of coordinate geometry, particularly studying lines, circles and conics.

1. An extra ingredient

Recall that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the set of ordered pairs (x, y) where $x, y \in \mathbb{R}$. We can visualise the points in this set by imagining them to live in a coordinate plane, as illustrated in figure 1. Notice that this visualisation tacitly adds something to $\mathbb{R}^2$

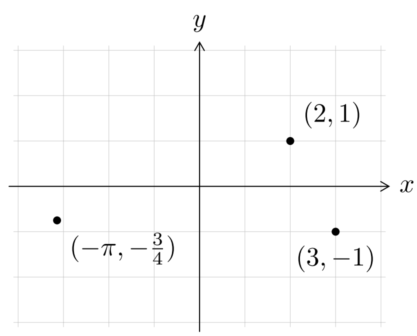

FIGURE 1. Points in the plane

which our formalisation using sets is lacking: a notion of *distance*. Indeed, we see (visually) that $(2, 1)$ is closer to $(3, -1)$ than to $(-\pi, -\frac{3}{4})$, but nothing about the set $\mathbb{R}^2$ allows us to deduce this formally without adding something else.

This something else is called a *metric* or a *distance function*.

Definition 1.1 (Metric Space)**.** Let M be a set, and let $d \colon M \times M \to \mathbb{R}$ be a function satisfying

I. $d(x, y) = 0$ if and only if $x = y$, (IDENTITY OF INDISCERNIBLES)
II. $d(x, y) = d(y, x)$, and (SYMMETRY)
III. $d(x, z) \leqslant d(x, y) + d(y, z)$, (TRIANGLE INEQUALITY)

for all $x, y, z \in M$. Then the function d is said to be a *metric* on M, and the pair (M, d) is said to be a *metric space*.

A metric space is any set M in which we can compare distances between pairs of points using a distance function d.

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Examples 1.2. (i) The real numbers with the metric

$$d(x, y) = |x - y|,$$

for instance, we have $d(2, 5) = 3$ and $d(-7, 3) = 10$. This metric just gives distances on the number line (see figure 2).

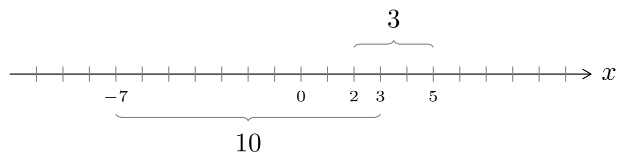

FIGURE 2. The metric $d(x, y) = |x - y|$ on the real line

(ii) The set of positive real numbers $(0, \infty)$ with the metric $d(x, y) = |\log(x/y)|$. In this metric, distances between numbers are measured multiplicatively instead of additively. For instance,

$$d(1, 3) = \log 3 = d(3, 9),$$

i.e., 3 is in the middle of 1 and 9. You can think of the number line with this metric getting denser and denser away from 1.

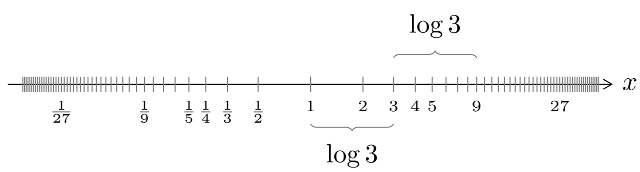

FIGURE 3. The metric $d(x, y) = |\log(x/y)|$ on the positive reals

(iii) The set of pairs of integer coordinates $\mathbb{Z} \times \mathbb{Z}$ with the metric

$$d\big((a, b), (c, d)\big) = |a - c| + |b - d|,$$

which counts the total vertical and horizontal displacement between points, as illustrated in figure 4. Notice that multiple "paths" give us the same distance.[1]

(iv) The set E of words in the English language, paired with the Levenshtein metric, defined as follows:

$$d(w_1, w_2) = \begin{array}{l}\text{the smallest number of insertions, deletions or}\\ \text{substitutions we can apply to } w_1 \text{ to get } w_2.\end{array}$$

For instance, the word kitten can become sitting by the following sequence of 3 edits:

<u>k</u>itten → <u>s</u>itten	(substitute k → s)
sitt<u>e</u>n → sitt<u>i</u>n	(substitute e → i)
sittin_ → sittin<u>g</u>	(insert g)

[1] In fact, if $h = |a - c|$ and $v = |b - d|$, then $d\big((a, b), (c, d)\big) = h + v$, and there are precisely $(h + v)!/(h!v!)$ different possible paths one can take from (a, b) to (c, d) along the grid. Do you see why?

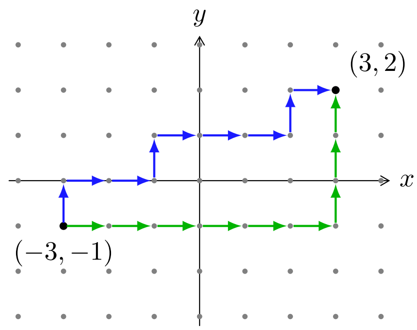

FIGURE 4. The "taxicab" metric on the lattice $\mathbb{Z} \times \mathbb{Z}$. The green and blue paths are two equivalent ways of computing the distance $d\big((-3,-1),(3,2)\big) = 9$.

It is not possible to achieve this with 2 or fewer edits, so their Levenshtein distance is $d(\text{kitten}, \text{sitting}) = 3$.

For each of these examples, make sure that you understand why:

(1) our chosen function introduces a notion of "distance" to the set in question,
(2) each of the metric functions satisfy properties I–III in definition 1.1,
(3) we would want a distance function to satisfy those properties.

It is productive to study metric spaces abstractly, i.e., it is useful to see what we can say about any general set M and function d which obey the properties I–III. This is because, if we prove something about any general metric space, it automatically applies to anything which fits the definition of a metric space, and as we have seen from examples 1.2, these can be wildly different from each other.

A trivial example of this: it makes sense that we would want "distance" to be non-negative; i.e., we want that $d(x,y) \geqslant 0$ for all $x, y \in M$. But this actually follows immediately from I–III, since

$$\begin{aligned} d(x,y) &= \tfrac{1}{2}\big(d(x,y) + d(x,y)\big) \\ &= \tfrac{1}{2}\big(d(x,y) + d(y,x)\big) && \text{(by II)} \\ &\geqslant \tfrac{1}{2}\, d(x,x) && \text{(by III)} \\ &= 0 && \text{(by I)}. \end{aligned}$$

This means that if we define a metric on a set, we don't need to bother checking it is always positive each time, since it if satisfies I–III, it inherits this from our reasoning above.

There are countless books on metric spaces, they are very useful things to study. We will not talk more about them here, we just simply point out that the introduction of a notion of distance is an extra step which we need to do ourselves in order to reason geometrically. Speaking of which, we haven't defined the metric we will be using for the rest of this chapter. Here it is:

Definition 1.3 (Euclidean Metric on $\mathbb{R}^2$)**.** Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be two points in $\mathbb{R}^2$. Then the (*Euclidean*) *distance* between A and B is defined by

$$d(A,B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

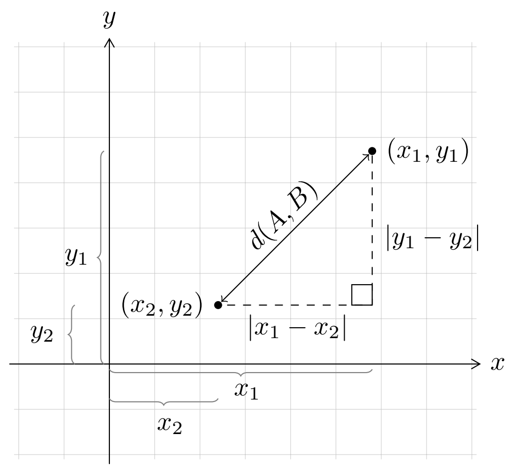

FIGURE 5. Points in the plane

Remark 1.4. This definition is inspired by Pythagoras' theorem. Indeed, if we have two points (x_1, y_1) and (x_2, y_2) in the plane, the horizontal displacement between them is $|x_1 - x_2|$ and the vertical displacement is $|y_1 - y_2|$.

Since the horizontal and vertical grid lines are perpendicular to each other, we get a right-angled triangle (see figure 5), and so Pythagoras' theorem gives us that

$$d(A, B)^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2,$$

and then taking square roots, we obtain definition 1.3.

Notice that we are appealing to visual intuition in our arguments here, relying on notions such as "parallel" and "right-angle" which we have not yet defined in our formal framework of sets. This is why we are saying that Pythagoras' theorem "inspires" the definition, because it is not really a theorem (in the sense that, we don't have a formal framework within which to prove it yet).

But what is a bit worrying is that our definition of distance seemingly embeds Pythagoras' theorem within it. Indeed, it is a philosophically challenging idea to grasp that we are *defining* distance here. In other words, in our formalism, Pythagoras' theorem is just an immediate consequence of the definition of distance. The deeper question here seems to be, why did God choose to use the Euclidean metric when it comes to measuring distances on a piece of paper?

If you are interested in going deeper down this rabbit hole, you can read this question that I asked myself a while ago on the Mathematics Stack Exchange website.

Notation (Delta prefix)**.** If we are working with a pair of subscripted variables, say, γ_1, γ_2, we denote the difference between them by $\Delta\gamma$, without the subscript. Moreover, $\Delta\gamma^2$ will mean $(\Delta\gamma)^2$, not $\Delta(\gamma^2)$.

This allows us to write the distance formula more concisely as $\sqrt{\Delta x^2 + \Delta y^2}$.

Example 1.5. The distance between $(1, 2)$ and $(5, -3)$ is

$$\sqrt{(1-5)^2 + (2 - -3)^2} = \sqrt{4^2 + 5^2} = \sqrt{41} \text{ units.}$$

Although we did say that we would not speak more about metric spaces, we ought to at least show that definition 1.3 defines a valid metric. Before we can do that however, we will need the following famous inequality for the last step of the proof:

Proposition 1.6 (Cauchy–Schwarz inequality)**.** *Let* $r_1, r_2, s_1, s_2 \in \mathbb{R}$. *Then*

$$(r_1s_1 + r_2s_2)^2 \leqslant ({r_1}^2 + {r_2}^2)({s_1}^2 + {s_2}^2).$$

Proof. Simply observe that if we expand the difference

$$({r_1}^2 + {r_2}^2)({s_1}^2 + {s_2}^2) - (r_1s_1 + r_2s_2)^2,$$

some terms cancel and we end up with

$${r_1}^2{s_2}^2 - 2r_1s_1r_2s_2 + {r_2}^2{s_1}^2.$$

Comparing this with the familiar $x^2 - 2xy + y^2 = (x-y)^2$, we see that it equals

$$(r_1s_2 - r_2s_1)^2,$$

which must be non-negative, and this completes the proof. □

Now we can prove that our definition of distance is a valid metric.

Theorem 1.7 ($\mathbb{R}^2$ is a metric space)**.** *Let* $A, B, C \in \mathbb{R}^2$. *Then the following properties hold:*

I. $d(A,B) = 0$ *if and only if* $A = B$, (Identity of Indiscernibles)
II. $d(A,B) = d(B,A)$, *and* (Symmetry)
III. $d(A,C) \leqslant d(A,B) + d(B,C)$, (Triangle Inequality)

Proof. Write $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$. For I, observe that

$$\begin{aligned} d(A,B) = 0 &\iff \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} = 0 \\ &\iff (x_1-x_2)^2 + (y_1-y_2)^2 = 0 \\ &\iff x_1 - x_2 = 0 \quad \text{and} \quad y_1 - y_2 = 0 \\ &\iff x_1 = x_2 \quad \text{and} \quad y_1 = y_2 \\ &\iff A = B. \end{aligned}$$

Next for II, we see that

$$d(A,B) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = d(B,A).$$

Finally for the triangle inequality, write $r_x = x_1 - x_2$, $r_y = y_1 - y_2$, $s_x = x_2 - x_3$ and $s_y = y_2 - y_3$. Then we want to prove that

$$\sqrt{(r_x+s_x)^2 + (r_y+s_y)^2} \leqslant \sqrt{{r_x}^2 + {r_y}^2} + \sqrt{{s_x}^2 + {s_y}^2},$$

and since both sides are non-negative, it's equivalent to proving that

$$(r_x+s_x)^2 + (r_y+s_y)^2 \leqslant {r_x}^2 + {r_y}^2 + {s_x}^2 + {s_y}^2 + 2\sqrt{{r_x}^2 + {r_y}^2}\sqrt{{s_x}^2 + {s_y}^2},$$

which simplifies to

$$r_xs_x + r_ys_y \leqslant \sqrt{{r_x}^2 + {r_y}^2}\sqrt{{s_x}^2 + {s_y}^2}.$$

If the left-hand side is negative, then the inequality is obviously true, so we may assume it is non-negative, so that we can square both sides and equivalently prove that

$$(r_x s_x + r_y s_y)^2 \leqslant ({r_x}^2 + {r_y}^2)({s_x}^2 + {s_y}^2).$$

But we did prove this, it's the Cauchy–Schwarz inequality! □

The proof of the triangle inequality isn't so straightforward, but what it tells us is intuitive: that the distance between the points A and C is less than the combined distance from A to B and then from B to C (figure 6). In fact, it gives

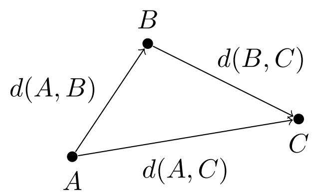

FIGURE 6. The triangle inequality

us a restriction on the side lengths of a triangle: the length of one side of a triangle cannot exceed the sum of the lengths of the other two.

Exercise 1.8.

1. For each of the following pairs of points in $\mathbb{R}^2$, determine the distance between them.
   a) $(1, 2)$ and $(3, 4)$ b) the origin and $(-3, 2)$
   c) $(-1, -3)$ and $(-7, 5)$ d) $(1, 2)$ and $(1, -2)$
   e) $(7, 4)$ and $(\pi, -2)$ f) (a, b) and $(-b, -a)$
2. Which of the following triples of numbers can be side lengths of a triangle?
   a) $2, 3, 4$ b) $4, 4, 8$ c) $5, 12, 13$
   d) $1, 2, 3$ e) $3, 5, 7$ f) $2, 5, 8$
3. Determine a condition on the value of a if the following are side lengths of triangles.
   a) $a, a + 1, a + 2$ b) a, a^2, a^3
4. Using the Cauchy–Schwarz inequality, prove that

   $$a + b \leqslant \sqrt{2}\sqrt{a^2 + b^2}.$$

   Hence or otherwise, deduce that for any $x, y > 0$,

   $$\sqrt{\frac{x}{x+y}} + \sqrt{\frac{y}{x+y}} \leqslant \sqrt{2}.$$

   Is this bound sharp? (i.e., is there an assignment of x and y for which we have equality instead of $\leqslant$?)

2. Midpoints and Lines

Intuitively, the midpoint of two other points is a point "in the middle" of them. We define it as follows.

Definition 2.1 (Midpoint)**.** Let $A = (x_1, y_1), B = (x_2, y_2) \in \mathbb{R}^2$. The *midpoint* or *centroid* of A and B is the point defined by

$$M = \Big(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\Big).$$

Let us start by proving that our definition of the midpoint exhibits the desired behaviour of being "in the middle" of A and B:

Theorem 2.2. *Let $A, B \in \mathbb{R}^2$, and let M be their midpoint. Then*

$$d(A, M) = d(M, B).$$

Proof. Write $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Then $M = \big(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\big)$, and we have

$$\begin{aligned} d(A, M) &= \sqrt{\Big(x_1 - \frac{x_1 + x_2}{2}\Big)^2 + \Big(y_1 - \frac{y_1 + y_2}{2}\Big)^2} \\ &= \sqrt{\Big(\frac{x_1 - x_2}{2}\Big)^2 + \Big(\frac{y_1 - y_2}{2}\Big)^2} \\ &= \sqrt{\Big(\frac{x_1 + x_2}{2} - x_2\Big)^2 + \Big(\frac{y_1 + y_2}{2} - y_2\Big)^2} = d(M, B), \end{aligned}$$

as required. □

Remark 2.3. Although the midpoint satisfies this condition, it is not a characterisation, i.e., there are other points which satisfy this. Indeed, pick any $\lambda \in \mathbb{R}$, then the point

$$N_\lambda = \Big(\frac{x_1 + x_2}{2} + \lambda(y_1 - y_2), \frac{y_1 + y_2}{2} - \lambda(x_1 - x_2)\Big)$$

also obeys $d(A, N_\lambda) = d(N_\lambda, B)$. In fact, the set of points

$$\{P \in \mathbb{R}^2 : d(A, P) = d(P, B)\}$$

for fixed $A, B \in \mathbb{R}^2$ turns out to be the *perpendicular bisector* of the *line segment* joining A and B. We have not defined any of these terms yet, but you should have an idea of what these terms mean intuitively. An equivalent way to describe it is as

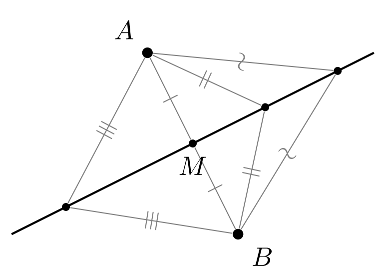

FIGURE 7. The set of points satisfying the equation $d(A, P) = d(P, B)$.

the set of all possible apices of isosceles triangles with base AB.[2] What is special about the midpoint is that, in addition to satisfying $d(A, M) = d(M, B)$, it also lies on the line segment joining A and B, as can be seen in figure 7.

[2]The *apex* (pl. *apices*) of an isosceles triangle is the vertex out of which the two sides of equal length emanate.