Online Maths Lessons

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october mmxix – march mmxx

Contents

1 Differential Equations (Continuation)

Consider the differential equation

$$\frac{dy}{dx} = f$$

or, written differently, $Dy = f$, where D represents differentiation of the function y.

To solve this specific type of differential equation, we just do $y = \int f\,dx$. We should think of $\int$ as D^{-1}, in a similar way to matrices. But what about the $+c$ thing?

The detail here is that D is not what we call "one-to-one" or "injective", i.e., it maps different inputs to the same output (e.g. x^2 is not injective because $(-2)^2 = (2)^2$ for example). In the case of D, we have $D[x^2+2] = D[x^2+3] = 2x$.

In the general case, it's difficult to study things which are not injective in some concrete way. But differentiation is a *linear* operation! We say that a function/operation/whatever T is linear if it obeys these two properties:

$$T(f+g) = T(f) + T(g) \qquad \text{and} \qquad T(\alpha f) = \alpha T(f),$$

where α is some constant, and f, g are "objects" which make sense in the context of T (if T is a matrix, they are vectors, if T is differentiation, they are functions).

Examples of functions/transformations which are linear:

- $f(x) = 2x$ is linear, $f(x+y) = 2(x+y) = 2x + 2y = f(x) + f(y)$ and $f(\alpha x) = 2\alpha x = \alpha(2x) = \alpha f(x)$.
- All functions defined by matrices are linear, e.g. $f(\mathbf{v}) = \mathbf{M}\mathbf{v}$ where $\mathbf{M}$ is some compatible matrix, then by matrix arithmetic $\mathbf{M}(\mathbf{x}+\mathbf{y}) = \mathbf{M}\mathbf{x}+\mathbf{M}\mathbf{y}$, and similarly for constants.
- We also know differentiation is linear, e.g. $\frac{d}{dx}(5x^2 + 2e^x) = 10x + 2e^x = 5\frac{d}{dx}(x^2) + 2\frac{d}{dx}(e^x)$.
- $\int$ is linear, $\sum$ is linear, etc.

The study of linear operations is called *linear algebra*.

Refer to the remark on page 3 of `https://drmenguin.com/files/odes.pdf`. The parts below (before the example) were written whilst we were reading through that remark.

$D[2f + 3g] = 2\cos(\,\cdot\,) + 6(\,\cdot\,) = 2D[f] + 3D[g]$

By something like $(3f)$ we mean $(3f)(x) = 3f(x)$, e.g. the function $3\sqrt{\cdot}$ just means the function $(3\sqrt{\cdot})(x) = 3\sqrt{x}$.

So in general, when we do something like $f + g$, we mean the function defined by $(f + g)(x) = f(x) + g(x)$ for all x, or αf means $(\alpha f)(x) = \alpha f(x)$ for all x.

E.g. $(3\cos + 2\sin)(x) = 3\cos x + 2\sin x$.

Matrix times the zero vector is zero. Why? Because $\mathbf{0} = 0\mathbf{0}$. Therefore $\mathbf{M}(\mathbf{0}) = \mathbf{M}(0\mathbf{0}) = 0\mathbf{M}(\mathbf{0}) = \mathbf{0}$. In conclusion, $\mathbf{M}(\mathbf{0}) = \mathbf{0}$.

Antiderivatives always differ by a constant, since $\ker(D)$ = the set of constant functions. This is not always immediately obvious like x and $x+3$, for example, $\int \sin x \cos x \, dx$ can either work out to $-\frac{1}{4}\cos 2x$ (by trig identity for $2\sin x$) or to $-\frac{1}{2}\cos^2 x$ by substition. Look at the two plots: https://www.desmos.com/calculator/czlz0rkt7g. You should be able to find a constant c such that $-\frac{1}{4}\cos 2x = -\frac{1}{2}\cos^2 x + c$ (by making use of some trig identity).

Example. We solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \cos 2x$.

(First we want to find the kernel of the operator defined by the LHS. We know that all functions in the kernel are of the form e^{kx} or some variant of this from the table in the notes).

Consider the homogeneous ($= 0$) case to determine the kernel.

Auxiliary equation: $k^2 - 5k + 6 = 0 \implies k = 2, 3$.

Thus the complentary function is $\mathrm{cf}(x) = c_1 e^{2x} + c_2 e^{3x}$.

(So what we have found here is the kernel of the operator $L = D^2 - 5D + 6I$. Any function of the form $c_1 e^{2x} + c_2 e^{3x}$ is mapped to $\mathbf{0}$ by L)

Now we try to guess a solution. We take $\mathrm{ts}(x) = \lambda \cos 2x + \mu \sin 2x$.

We work out the derivatives to substitute in the lhs:

$\mathrm{ts}'(x) = -2\lambda \sin 2x + 2\mu \cos 2x$
$\mathrm{ts}''(x) = -4\lambda \cos 2x - 4\mu \sin 2x$

So we want

$$\begin{aligned}
\mathrm{ts}'' - 5\,\mathrm{ts}'(x) + 6\,\mathrm{ts}(x) &= \cos 2x \\
(-4\lambda \cos 2x - 4\mu \sin 2x) - 5(-2\lambda \sin 2x + 2\mu \cos 2x) + 6(\lambda \cos 2x + \mu \sin 2x) &= \cos 2x \\
(-4\lambda - 10\mu + 6\lambda)\cos 2x + (-4\mu + 10\lambda + 6\mu)\sin 2x &= \cos 2x \\
(2\lambda - 10\mu)\cos 2x + (2\mu + 10\lambda)\sin 2x &= \cos 2x
\end{aligned}$$

so we solve simultaneously

$$\begin{cases} 2\lambda - 10\mu = 1 & \text{eqn. (1)} \\ 2\mu + 10\lambda = 0 & \text{eqn. (2)} \end{cases}$$

$(1) + 5(2) \implies 52\lambda = 1 \implies \lambda = \frac{1}{52}$, and then by (2), $\mu = -\frac{5}{52}$.

(Here we've managed to find a particular function pi(x) which, when plugged into the lhs, yields the rhs. Thus, all solutions are of the form $y(x) = \text{pi}(x) + \text{cf}(x)$).

Therefore general solution is $\boxed{y(x) = \frac{1}{52}(\cos 2x - 5\sin 2x) + c_1 e^{2x} + c_2 e^{3x}}$.

Now say we were given that $y(0) = y'(0) = 1$.

$y(0) = 1 \implies \frac{1}{52}(1) + c_1 + c_2 = 1 \implies c_2 = \frac{51}{52} - c_1 \quad (*)$.

Also $y'(x) = \frac{1}{52}(-2\sin 2x - 10\cos 2x) + 2c_1 e^{2x} + 3c_2 e^{3x}$, so then

$y'(0) = 1 \implies \frac{1}{52}(0 - 10) + 2c_1 + 3c_2 = 1$
$\implies 2c_1 = \frac{31}{26} - 2c_2 = \frac{31}{26} - 3(\frac{51}{52} - c_1)$ by $(*)$

which gives that $c_1 = \frac{7}{4}$, and then by $(*)$, $c_2 = \frac{51}{52} - \frac{7}{4} = \frac{10}{13}$. Thus the **particular solution** is

$\boxed{y(x) = \frac{1}{52}\left(\cos 2x - 5\sin 2x + 91e^{2x} + 40e^{3x}\right)}$.

LESSON 2 *23rd November, 2019*

2 Integration Techniques and Numerical Methods

So far, we have seem some analytic methods to evaluate integrals. But some integrals cannot be "done" this way, a simple example is the indefinite integral

$$\int e^{x^2}\, dx,$$

which cannot be written down using elementary functions.

There is some subtlety here, as to what we mean by "done", we usually mean a closed-form solution (https://en.wikipedia.org/wiki/Closed-form_expression). The reason this is subtle is because a lot of functions are *defined* by integrals. If you ever look into the formal definition of $\ln x$, it is actually

$$\ln(x) = \int_1^x \frac{1}{t}\, dt,$$

which in some sense is cheating, because rather than working out $\int \frac{1}{t}\, dt$, we have just given it a name and made use of that name in the answer. So here we could analogously "cheat" and define a new function $\text{lm}(x) = \int_0^x e^{t^2}\, dt$, and if you differentiate $\text{lm}(x)$, by the fundamental theorem of calculus, its derivative is e^{x^2}, so $\int e^{x^2}\,dx = \operatorname{lm}(x) + c$. There is actually a function defined this way, but it's called the imaginary Gaussian error function $\operatorname{erfi}(x)$ instead of $\operatorname{lm}(x)$: https://en.wikipedia.org/wiki/Error_function (it is slightly different from $\operatorname{lm}(x)$ in that it has a factor of $\frac{2}{\sqrt{\pi}}$).

2.1 Extract from notes about integration (just gloss through these)

How can we attempt to formalise the integral? Let's try to find the area bounded by the curve $y = x^2$ between $x = 0$ and $x = b \in \mathbb{R}$ using only the elementary notion of the area of a rectangle.

Let us subdivide the interval $[0, b]$ into n subintervals of equal length:

$$\left[0, \frac{b}{n}\right], \left[\frac{b}{n}, \frac{2b}{n}\right], \left[\frac{2b}{n}, \frac{3b}{n}\right], \dots, \left[\frac{(n-1)b}{n}, \frac{b}{n}\right]$$

and superimpose rectangles on top of each interval, each of which has height equal to the highest point of the curve within that interval. In the case of $y = x^2$, this point always happens to be the end of the interval (since x^2 is increasing).

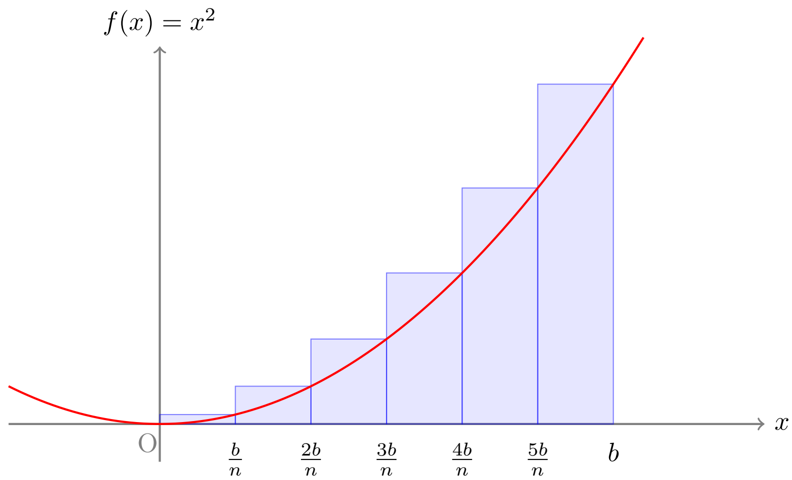


Illustration of upper-rectangles with $n = 6$.

We will call these rectangles **upper-rectangles**, since the curve will always be trapped below them. Now consider the area of the first rectangle, which we will denote A_1:

$$A_1 = \text{height} \times \text{width} = f\left(\frac{b}{n}\right) \cdot \frac{b}{n} = \frac{b^2}{n^2} \cdot \frac{b}{n} = \frac{b^3}{n^3}.$$

Similarly, we obtain the area of the second rectangle, A_2:

$$A_2 = f\left(\frac{2b}{n}\right) \cdot \frac{b}{n} = 4\frac{b^3}{n^3}.$$

In general, the area of the ith rectangle A_i is

$$A_i = f\left(\frac{ib}{n}\right) \cdot \frac{b}{n} = i^2 \frac{b^3}{n^3}.$$

Now if we determine the sum of all the A_i's, we will obtain an *overestimate* of the desired area. Let us determine this overestimate:

$$\begin{aligned} A_U = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} i^2 \frac{b^3}{n^3} = \frac{b^3}{n^3} \sum_{i=1}^{n} i^2 &= \frac{b^3}{n^3} \cdot \frac{n}{6}(n+1)(2n+1) \\ &= \frac{b^3}{6n^2}(2n^2 + 3n + 1) = \frac{b^3}{3}\left(1 + \frac{3}{2n} + \frac{1}{2n^2}\right). \end{aligned}$$

Let us now try and determine an *underestimate* for the desired area, this time by considering what we shall call **lower-rectangles**. We construct these over the subintervals in a similar way to the upper-rectangles, however this time their height is the *lowest* point of the curve within the interval:

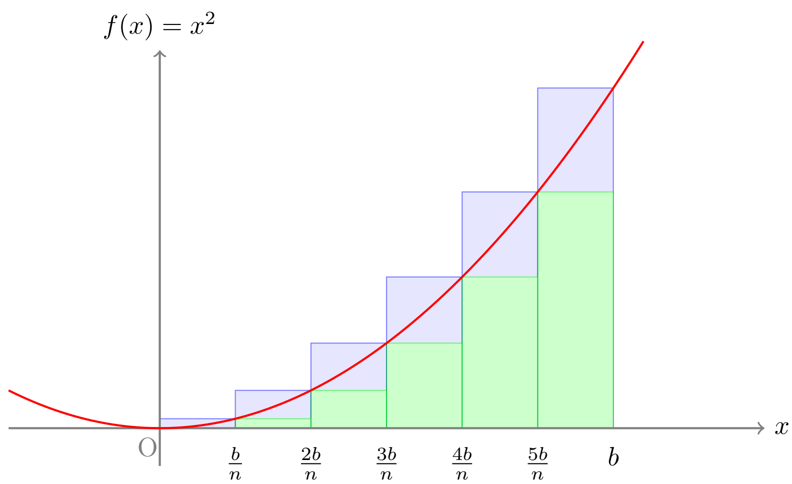


Illustration of lower-rectangles with $n = 6$.

Now the area of the first lower rectangle, which we will denote a_1, is given by

$$a_1 = \text{height} \times \text{width} = f(0) \cdot 0 = 0,$$

the second lower rectangle has area a_2, given by

$$a_2 = f\left(\frac{b}{n}\right) \cdot \frac{b}{n} = \frac{b^2}{n^2} \cdot \frac{b}{n} = \frac{b^3}{n^3}.$$

In general, the area a_i of the ith rectangle is

$$a_i = f\left(\frac{(i-1)b}{n}\right) \cdot \frac{b}{n} = (i-1)^2 \frac{b^3}{n^3}.$$

We can find the sum of areas of all these rectangles, just as we did with the upper rectangles:

$$A_L = \sum_{i=1}^{n} a_i = \frac{b^3}{n^3} \sum_{i=1}^{n} (i-1)^2 = \frac{b^3}{6n^2}(n-1)(2n-1) = \frac{b^3}{3}\left(1 - \frac{3}{2n} + \frac{1}{2n^2}\right)$$

which is an underestimate.

Now the desired area, A, is clearly bounded above by A_U and below by A_L, i.e., we have

$$A_L \leqslant A \leqslant A_U$$

i.e.,

$$\frac{b^3}{3}\left(1 - \frac{3}{2n} + \frac{1}{2n^2}\right) \leqslant A \leqslant \frac{b^3}{3}\left(1 + \frac{3}{2n} + \frac{1}{2n^2}\right)$$

and as we take more rectangles (i.e., split our interval $[0, b]$ into narrower subintervals), we see that A_L and A_U converge towards the value $b^3/3$. In fact, using the definition of the limit of a sequence,[1] we can treat A_L and A_U as sequences in n, the number of rectangles, and confirm that

$$\lim_{n\to\infty} A_L = \frac{b^3}{3} \qquad \text{and} \qquad \lim_{n\to\infty} A_U = \frac{b^3}{3}.$$

Thus we have established that our area A is sandwiched between A_L and A_U for any number of rectangles n, and that the sequences (A_L) and (A_U) converge to the same limit. If two sequences (a_n) and (b_n) converge to the same limit a and $a_n \leqslant c \leqslant b_n$ for all n, then $a = c$,[2] so it makes sense to define the area to be this limit, i.e., $A = b^3/3$.

This is called Riemann integration—it is defined only when A_U and A_L are close for large n. If this is not the case, the integral is not defined. An example:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise.} \end{cases}$$

Since "most numbers" are irrational, it makes sense that $\int_0^1 f(x)\,dx$ should equal 0. But Riemann always gives that $A_U = 1$ and $A_L = 0$, so we cannot integrate this using Riemann integration. A "stronger" version, called Lebesgue integration, is used.

[1] https://en.wikipedia.org/wiki/Limit_of_a_sequence

[2] https://en.wikipedia.org/wiki/Squeeze_theorem#Statement

2.2 Trapezium Rule

Here we try to approximate integrals using methods inspired by the definition of the Riemann integral.

Page 12 of booklet: https://maths.com.mt/assets/files/booklet.pdf

Example of Trapezium rule: $\int_0^1 e^{x^2}\,dx$ using 4 ordinates, answer accurate to 4 d.p.s.

First, we need h:

$$h = \frac{1-0}{3} = \frac{1}{3}.$$

Then we work out the y_i's (we work with 6 d.p.s so the answer is definitely accurate to 4 d.p.s):

x_i	0	$\frac{1}{3}$	$\frac{2}{3}$	1
y_i	1	1.117519	1.559623	2.718282

Therefore $\int_0^1 e^{x^2}\,dx \approx \frac{1}{6}(1 + 2.718282 + 2(1.117519 + 1.559623)) = 1.5121.$

The "actual" answer is ~ 1.46265. This is what we did: https://www.desmos.com/calculator/yikw2vzd3b

2.3 Simpson's Rule

Instead of joining two points with a line (which is what is done in the trapezium rule) we can use a quadratic. The different ways of joining two points is called *interpolation*, e.g. Bezier curves: https://www.jasondavies.com/animated-bezier/. Here is a example of how Simpson's rule uses quadratic interpolation to better approximate a curve than with lines: https://upload.wikimedia.org/wikipedia/commons/6/67/Simpsonsrule2.gif.

You always need an even number of strips (i.e., an odd number of ordinates), so we have $y_0, y_1, \ldots, y_n$ where n is even. The formula is

$$A = \int_a^b y\,dx \approx \frac{h}{3}(y_0 + y_n + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})).$$

We do the same example, this time with Simpson's rule, and we compare.

Let's do $\int_1^5 \frac{x^3}{\sqrt{1+x}}\,dx$.

a) Approximate the area using the Trapezium Rule with 5 ordinates.

b) Do the same thing with Simpson's rule.

c) Calculate the integral analytically and find the percentage error in both of your calculations.

a) We have $h = \frac{5-1}{4} = 1$. Therefore

	x_0	x_1	x_2	x_3	x_4
x_i	1	2	3	4	5
y_i	0.707107	4.618800	13.500000	28.621700	51.031000

Using the trapezium rule,

$$\begin{aligned} A &\approx \frac{1}{2}(0.707107 + 51.031 + 2(4.6188 + 13.5 + 28.6217)) \\ &= 72.6096 \end{aligned}$$

Plot: https://www.desmos.com/calculator/ttbmpwcjd5

b) Using Simpson's rule, we have

$$\begin{aligned} A &\approx \frac{1}{3}(0.707107 + 51.031 + 4(4.6188 + 28.6217) + 2(13.5)) \\ &= 70.5667 \end{aligned}$$

c) Put $u = 1 + x$. Then $du = dx$, $x = 1 \Rightarrow u = 2$, and $x = 5 \Rightarrow u = 6$. Therefore the integral becomes

$$\int_2^6 \frac{(u-1)^3}{\sqrt{u}}\,du$$

Which expands out to

$$\int_2^6 (u^{5/2} - 3u^{3/2} + 3u^{1/2} - u^{-1/2})\,du = \frac{2}{35}(9\sqrt{2} + 499\sqrt{6}),$$

which is approximately 70.5728.

For percentage error, we do

$$\text{Percentage error} = \left|\frac{\text{Actual} - \text{Approximation}}{\text{Actual}}\right| \times 100\%.$$

So, for the trapezium rule, we have $\frac{70.5728-72.6096}{70.5728} = 2.9\%$. On the other hand, $\frac{70.5728-70.5667}{70.5728} = 0.008\%$.

Note. In general, Simpson's rule produces much better results than the trapezium rule.

3 Lessons in Person (Malta)

At this stage we met up in Malta and covered, in person, over two lessons, the following topics:

- **Newton–Raphson Method**
  This is used to improve a guess x_0 of a root of $f(x)$ by iterating the formula in the booklet.

- **Integration by Reduction**
  Formulas with I_n. Usually use these to reduce integrals like $\int \cos^n x \, dx$ in terms of simpler ones.

- **Length of Arc and Volume of Revolution**
  Just formulae from the booklet, we did a few examples.

Usually there is a whole question on numerical methods (i.e., some integral which you find using Simpson's/Trapezium, together with some Newton–Raphson problem). Then there is another one with integration by reduction and length of arc. Take a look at past papers to confirm this, it is *usually* the case but not guaranteed.

We also started polar coordinates, and we covered these topics in that chapter:

- **Drawing Polar Curves**
  We drew some polar curves by evaluating them at the "special angles" (constructing a table, etc). We noticed that some special curves have symmetry, depending on their form. We discussed special curves like cardioids (and more generally limaçon curves), as well as roses, lines.

- **The Relationship between Polar and Cartesian Coordinates**
  We saw how to swap between curves in x and y to curves in r and θ and vice-versa. An example we did was to express $y = x^2$ as $r = \sec\theta \tan\theta$, and $r = 1 + 2\cos\theta$ as $x^2 + y^2 = (x^2 + y^2 - 2x)^2$.

  In general for these it is useful to know how to find $\cos\theta$ and $\sin\theta$ when $\tan\theta = y/x$ using trigonometry trickery.

- **Polar Integration**
  We solved problems on finding areas bounded by polar curves. This basically involves using a formula from the booklet.

The only thing we had left to cover were tangents to a polar curve.

LESSON 3 *1st February, 2020*

4 Polar Coordinates

You asked me the following problem on polar integration.

Problem: Find the (smaller) area bounded between the curves $r = e^\theta$ and $r = \cos\theta$ (for $-\pi \leqslant \theta \leqslant \pi$).

First we need to find the point at which the two curves intersect, for the limits of integration: https://www.desmos.com/calculator/h3ceonosbe.

$\theta = 0$ is obvious, but the other is not easy to obtain analytically (most likely it is impossible to do so), so we use something like Newton–Raphson to get $\theta = -1.2927$.

So we do

$$\begin{aligned} A &= A_{\text{circle}} - A_{\text{exponential}} \\ &= \frac{1}{2}\int_{-1.2927}^{0} \cos^2\theta \, d\theta - \frac{1}{2}\int_{-1.2927}^{0} e^{2\theta} \, d\theta \\ &= 0.1580 \end{aligned}$$

4.1 Tangents

For these, it is best to follow the method described at the end of these notes: https://drmenguin.com/files/polar.pdf.

E.g. $r = 1 + 2\cos\theta$: https://www.desmos.com/calculator/tiagbrm9mw

For tangents at the pole, we solve $1 + 2\cos\theta = 0$ in the range $-\pi \leqslant \theta \leqslant \pi$.

$1 + 2\cos\theta = 0 \implies \cos\theta = -\frac{1}{2}$, $\theta_{\text{pv}} = \cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$, so $\theta = \pm\frac{2\pi}{3} + 2\pi n$ for $n \in \mathbb{Z}$. Within the range, we get $\theta = \frac{2\pi}{3}$ and $\theta = -\frac{2\pi}{3}$.

For horizontal tangents, we solve $\frac{dr}{d\theta} = -r\cot\theta$.

Notice $r = 1 + 2\cos\theta$, so $\frac{dr}{d\theta} = -2\sin\theta$, and we have

$$-2\sin\theta = -(1 + 2\cos\theta)\cot\theta.$$

Multiplying by $\sin\theta$ throughout,

$$2\sin^2\theta = \cos\theta + 2\cos^2\theta$$

and then using the Pythagorean identity, we get

$$2(1 - \cos^2\theta) = \cos\theta + 2\cos^2\theta,$$

which rearranges to

$$4\cos^2\theta + \cos\theta - 2 = 0,$$

so by completing the square, we get

$$\cos\theta = \frac{-1 \pm \sqrt{33}}{8},$$

and then either $\theta_{\text{pv}} = \cos^{-1}(\frac{-1+\sqrt{33}}{8})$ or $\theta_{\text{pv}} = \cos^{-1}(\frac{-1-\sqrt{33}}{8})$.

Using the general solution, we get the four solutions $\theta = \pm\cos^{-1}(\frac{-1+\sqrt{33}}{8}) \approx \pm 0.9359$, $\theta = \pm\cos^{-1}(\frac{-1-\sqrt{33}}{8}) \approx \pm 2.5738$.

We find the corresponding r-coordinate of the curve at each angle, which yields the points $(2.1862, \pm 0.9359)$, $(-0.6861, \pm 2.5738)$. These four points lie on the four horizontal tangents to the curve, and we need to find the appropriate value of k for these tangents.

$(2.1862, -0.9359) \implies 2.1862 = k\csc(-0.9359) \implies k = 2.1862\sin(-0.9359) = -1.7602$

$(2.1862, 0.9359) \implies 2.1862 = k\csc(0.9359) \implies k = 2.1862\sin(0.9359) = 1.7602$

$(-0.6861, -2.5738) \implies -0.6861 = k\csc(-2.5738) \implies k = -0.6861\sin(-2.5738) = -0.3689$

$(-0.6861, 2.5738) \implies -0.6861 = k\csc(2.5738) \implies k = -0.6861\sin(2.5738) = 0.3689$

Therefore the horizontal tangents are

$r = \pm 1.7602\csc\theta$ and $r = \pm 0.3689\csc\theta$.

For vertical tangents, we do an identical procedure but solve instead $\frac{dr}{d\theta} = r\tan\theta$. The solutions to this equation are $\theta = 0, \pi, \pm 1.8234$, which correspond to points $(3, 0)$, $(-1, \pm\pi)$, $(0.5, \pm 1.8234)$, and give the equations $r = 3\sec\theta$, $r = \sec\theta$ and $r = -0.1249\sec\theta$.

LESSON 4 *28th February, 2020*

5 Curve Sketching

5.1 Some Revision / Reflection

At this point, we already know how to sketch some curves given by both explicit and implicit functional equations. We either have things like

$$y = f(x),$$

e.g., $y = x^2$, $y = \cos(2x)$, etc, whose y-coordinate is given explicitly by a function. We also have the more general "implicit" equations, and these are of the form

$$F(x, y) = 0,$$

which tell us a condition which x and y must satisfy in order to be on the curve, but do not explicitly tell us how to find y given x, or vice-versa. E.g., the circle $x^2 + y^2 = 1$, or the hyperbola $xy = 1$. (In these two examples you can make y subject with some success, but in general it is not always possible, such as $x^2 + y^2 = (x^2 + y^2 - 2x)^2$, which is equivalent to the cardioid $r = 1 + 2\cos\theta$.)

Explicit curves we already know. We can sketch $y = f(x)$ when $f(x)$ is one of:

- x
- x^2
- $|x|$
- x^3
- e^x
- $\ln x$
- $\frac{1}{x}$
- $\cos x$
- $\sin x$
- $\tan x$

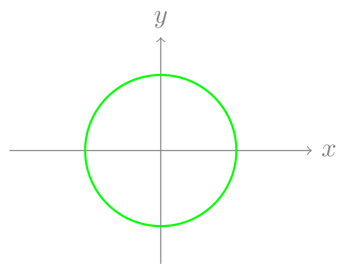
Also, if we know how to draw $y = f(x)$, then we also know how to draw

- $y = f(x) + k$ $(k \in \mathbb{R})$
- $y = f(x + k)$ $(k \in \mathbb{R})$
- $y = -f(x)$
- $y = f(-x)$
- $y = k\,f(x)$ $(k \in \mathbb{R})$
- $y = f(k\,x)$ $(k \in \mathbb{R})$

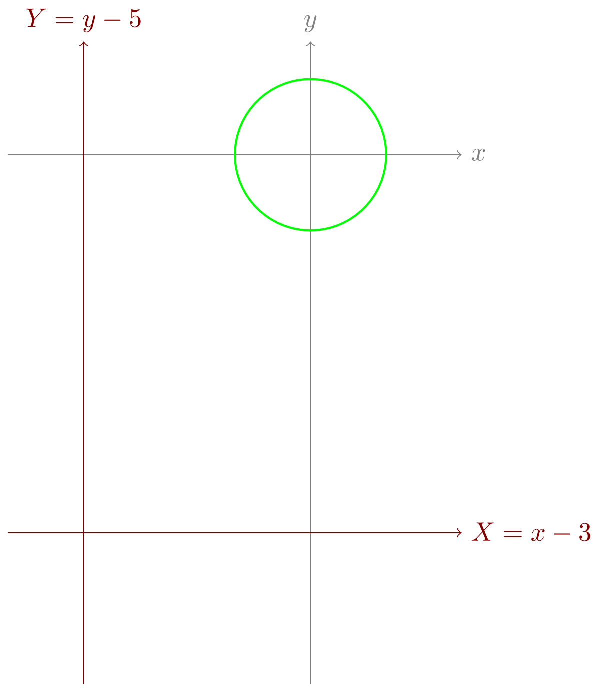
It is strange that, when we change the x-coordinate versus the y-coordinate, we get different behaviour. For example, adding to the x-coordinate moves the curve left (i.e., in the *negative* x-direction), whereas adding to the y-coordinate moves the curve up (i.e., in the *positive* y-direction). Similar for stretching/squishing when we multiply by a constant, the behaviour is inconsistent. Why?

We can understand these better if we properly understand the implicit case, $F(x, y) = 0$. Suppose we can draw $F(x, y) = 0$, how do we draw the following?

- $F(x + k, y + \ell) = 0$
  This is going to "translate" the axes. For example, take the circle $x^2 + y^2 = 1$. This is centred at the origin.



Let's say we want to move the circle so that the centre is now $(3,5)$. What is the equation describing this new curve? We can achieve this simply by shifting the y-axis downwards by 5 units, and the x-axis leftwards by 3 units. The corresponding equation is $F(x-3, y-5) = 0$, i.e., $(x-3)^2 + (y-5)^2 = 0$.



Alternatively, we can see it this way: we changed things so that $(-3,-5)$ is the "new" origin, and similarly $(x-3, y-5) = (X, Y)$ is the new (x, y).

How does this relate to the explicit case $y = f(x)$? Then the equation is

$$\underbrace{y - f(x)}_{F(x,y)} = 0,$$

so if we want to move a curve to the right by a units and up by b units, we do

$$F(x-a, y-b) = 0 \iff (y-b) - f(x-a) = 0 \iff y = f(x-a) + b.$$

This is why things appear inconsistent. The actual fact of the matter is, when we add to $f(x)$, we are not adding, but we are *subtracting* from the y-coordinate! ($y = f(x) + b$ is equivalent to $y - b = f(x)$.)

- $F(\pm x, \pm y) = 0$,
  this is similar to the explicit case, changing sign of x-coordinate "reverses" the direction of the x-axis, so we mirror the x-axis, and similarly for the y.

  For example, the circle $(x-1)^2 + (y-1)^2 = 1$. If we negate both coordinates, we get $(-x-1)^2 + (-y-1)^2 = 1$. This is the same as $(x+1)^2 + (y+1)^2 = 1$, i.e., the first circle reflected in both axes.

- $F(ax, by) = 0$.
  This is going to compress both axes, the x by a factor of a and the y by a factor of b.

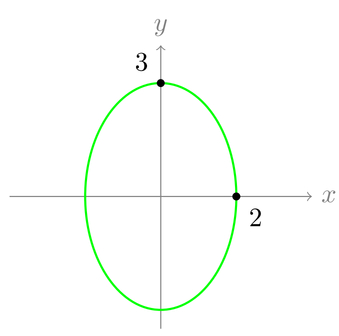
  Let's think of $F(x, y) := y - \cos(x) = 0$ (i.e., the graph of $y = \cos x$). A point on this curve (and a solution to $F(x, y) = 0$) is $(x_0, y_0) = (\frac{\pi}{3}, \frac{1}{2})$. If we have instead, $F(3x, 2y) = 2y - \cos(3x) = 0$, this point will correspond to $(\frac{\pi}{9}, \frac{1}{4})$, since then we get $F(3\frac{x_0}{3}, 2\frac{y_0}{2}) = F(x_0, y_0) = 0$.

  In the case $y = f(x)$, we have $F(x, y) = y - f(x) = 0$. Thus

  $$F(ax, by) = 0 \implies by = f(ax) \implies y = \tfrac{1}{b} f(ax).$$

  This is why in the implicit case, multiplication outside "stretches" rather than compresses.

  Example: draw $x^2/4 + y^2/9 = 1$. If we take $F(x, y) = x^2 + y^2 - 1$, we recognise that this is $F(x/2, y/3)$, so we draw this by compressing by a factor of $\frac{1}{2}$ (i.e., stretching by a factor of 2) the circle in the x-direction, and similarly for y.

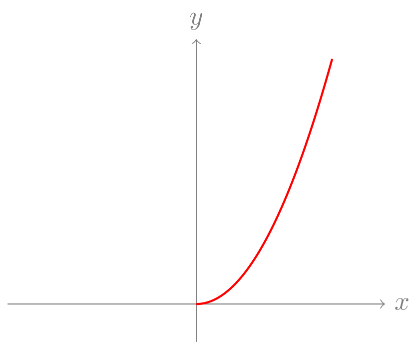


These three cases incorporate the six we had before, since in truth, there is no difference between how we change the x versus the y coordinate.

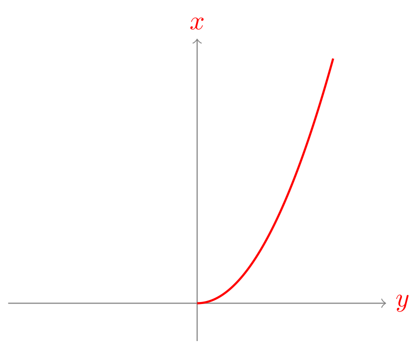
Now at this point in the lesson, you asked about:

The inverse of a function. The graph of an injective function f is the set of points (x, y) such that $f(x) = y$. The graph of the inverse of f is the set of points (x, y) such that $f(y) = x$, since the function f^{-1} is such that $f^{-1}(x) = y$ for every (x, y) satisfying $f(y) = x$. So if we set $F(x, y) := y - f(x)$, its graph is the set of points satisfying $F(x, y) = 0$, whereas the graph of the inverse is the set of points satisfying $F(y, x) = 0$. (This is a bit hard to read but if you read it slowly it should make sense.)

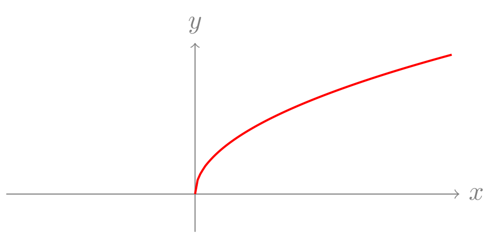
Example, Let f be the function $f\colon [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2$. This is injective.



The inverse is basically this:



This indeed is the set of points satisfying $F(y, x) = 0$. But we usually like to have the x-axis on the horizontal, so we can switch these by reflection in $y = x$:

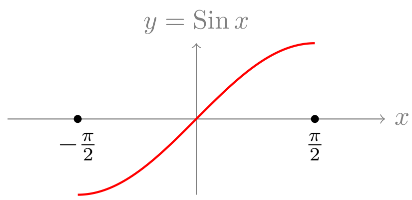


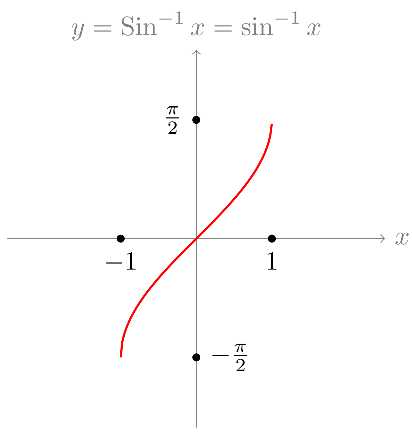
Inverse trigonometric functions. The trigonometric functions are not injections, therefore they have no inverse. So what are the "inverse trigonometric functions" we use on our calculator, say?

What we usually do when we have "inverses" of things which are not injections is we restrict the domain of the function so that they are injections. Then we call the inverse of the injections the inverses of the original function (and sometimes use the term "principal value").

E.g. Take $f(x) = x^2$ on $\mathbb{R}$. This is not invertible. But if we then take $F = f \restriction [0, \infty)$,[3] then we have an injection. The inverse is $\sqrt{x}$. We could call this the "principal value", of x (say we are solving $x^2 = 4$), and then the general solution is given by $x = \pm x_{\text{pv}}$.

This is what we do with sin, cos and tan. We set $\text{Sin} = \sin \restriction [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\text{Cos} = \cos \restriction [0, \pi]$ and $\text{Tan} = \tan \restriction [-\frac{\pi}{2}, \frac{\pi}{2}]$. We then denote Sin^{-1} by $\sin^{-1}$, and so on.





This function has domain $[-1, 1]$ and codomain $[-\frac{\pi}{2}, \frac{\pi}{2}]$

[3] Let $f : X \to Y$ be a function, and let $A \subseteq X$. Then the *restriction of* f *to* A is a function $g : A \to Y$ such that $g(x) = f(x)$ for all $x \in A$. We denote g by $f \restriction A$.

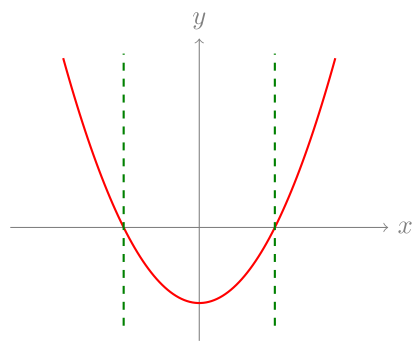
5.2 The Reciprocal Function

Suppose we know the graph of $y = f(x)$. The goal here is to sketch

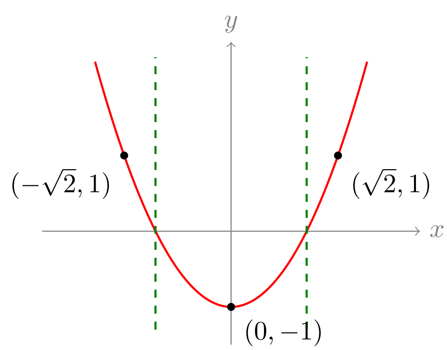
$$y = \frac{1}{f(x)}.$$

We can do this by considering the following, with the example $y = \frac{1}{x^2-1}$.

- If $f(x) = 0$, we get a vertical asymptote for $\frac{1}{f(x)}$. Conversely, if at $x = a$ there is an asymptote, then $\frac{1}{f(x)}$ has a root there. The asymptotes/intercepts divide the curve into parts which are each either completely above the x-axis or below.



- If within a part, $f(x) > 0$, then $\frac{1}{f(x)} > 0$, and similarly if $f(x) < 0$, then $\frac{1}{f(x)} < 0$, so the quadrants in which part of a curve appears do not change.
- If (x, y) is a point on $y = f(x)$, then $(x, \frac{1}{y})$ is a point on the reciprocal. So any points of the form $(x, \pm 1)$ remain unchanged by this transformation.

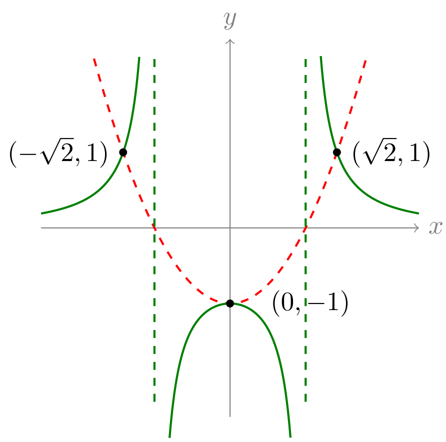


- We also have:

– If as $x \to a$ (where a is either $\pm\infty$ or the x-coordinate of an asymptote) $f(x) \to \infty$, then $\frac{1}{f(x)} \to 0^+$, and vice-versa.

– If as $x \to a$ (where a is either $\pm\infty$ or the x-coordinate of an asymptote) $f(x) \to -\infty$, then $\frac{1}{f(x)} \to 0^-$, and vice-versa.

With these considerations, we can draw:

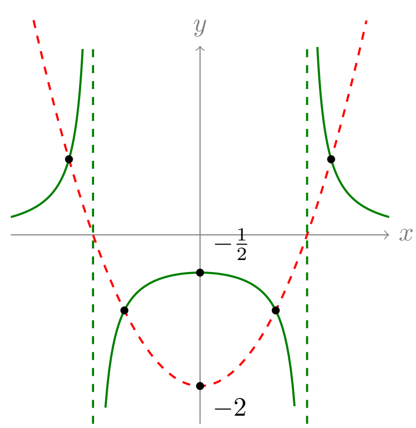


Notice that

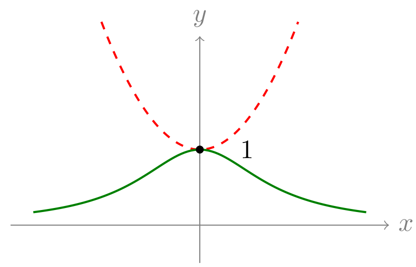
- If $f(x)$ has a y-intercept at $y = a$, then $y = \frac{1}{f(x)}$ has a y-intercept at $y = \frac{1}{a}$.
- If $f(x)$ has a maximum turning point at (x, y), then $y = \frac{1}{f(x)}$ has a minimum turning point at $(x, \frac{1}{y})$.

Let's now do $y = \frac{1}{x^2-2}$.

This is similar but the turning point is no longer fixed at -1, it goes from -2 to $-\frac{1}{2}$.



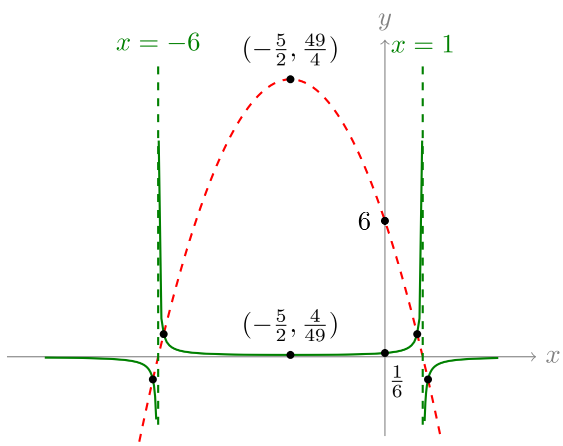
Next, we do $y = \frac{1}{x^2+1}$.



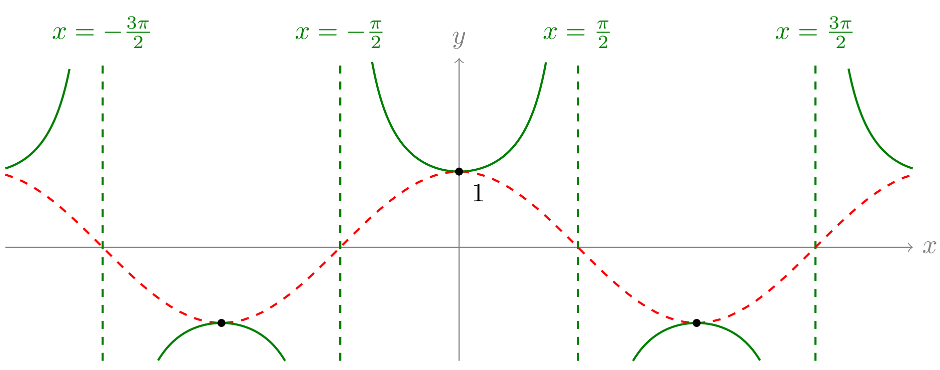
Another example, $y = \frac{1}{6-5x-x^2}$.

Notice $6 - 5x - x^2 = -(x^2 + 5x - 6) = -(x + 6)(x - 1)$.

The turning point occurs at $x = -b/2a = -5/2$, with y-coordinate $6 + 25/2 - 25/4 = 49/4 \approx 12$.



Another example, let us draw $y = \sec x$.



5.3 Rational Functions

We would like to sketch functions of the form

$$y = \frac{ax^2 + bx + c}{dx^2 + ex + f},$$

where $a, b, c, d, e, f \in \mathbb{R}$.

First of all, if we have

$$y = \frac{ax + b}{cx + d},$$

we can rewrite this using long division as

$$y = A + \frac{B}{cx + d}$$

where $A = a/c$ and $B = (bc - ad)/c$, and recognise that it is a series of transformations applied to $1/x$ which we can draw:

$$\frac{1}{x} \xrightarrow{f(x+d)} \frac{1}{x+d} \xrightarrow{f(cx)} \frac{1}{cx+d} \xrightarrow{B\,f(x)} \frac{B}{cx+d} \xrightarrow{f(x)+A} A + \frac{B}{cx+d}.$$

Remark. When combining these transformations more generally, it is often desired to obtain $f(ax + b)$. Notice that this corresponds to $f(x + b)$ followed by $f(ax)$, and not vice-versa! The other way around produces $f(a(x + b)) = f(ax + ab)$.

For example, say you want to draw $\sin(2x + \frac{\pi}{3})$. First, shift by $\frac{\pi}{3}$, and *then* compress things by a factor of 2. Doing things the other way around will give the graph of $\sin(2(x + \frac{\pi}{3})) = \sin(2x + \frac{2\pi}{3})$, which is not what we wanted.

Now suppose one of $a, d \neq 0$. There are two techniques.

The First Technique: Using Calculus

We need to consider 4 things to sketch these curves.

(i) The coordinates of any x- and y-intercepts.

(ii) The coordinate of any stationary points.

(iii) The equations of any linear asymptotes to the curve.

(iv) The behaviour of the curve as $x \to \pm\infty$.

An example:

$$y = \frac{2x^2 + 1}{(x + 1)(x - 2)}.$$

1. There are no x-intercepts since the numerator is never zero. The curve intersects the y-axis at $y = \frac{1}{(1)(-2)} = -\frac{1}{2}$.

2. Next, we differentiate and set the derivative equal to zero.

$$\begin{aligned} 4x(x+1)(x-2) - (2x^2+1)(2x-1) &= 0 \\ \implies x^2 + 5x - \tfrac{1}{2} &= 0 \\ \implies (x+\tfrac{5}{2})^2 - \tfrac{25}{4} - \tfrac{1}{2} &= 0 \\ \implies x = -\tfrac{5}{2} \pm \tfrac{3\sqrt{3}}{2}&. \end{aligned}$$

   These have corresponding y-coordinates $\frac{2}{3}(1 \pm \sqrt{3})$. Numerically, the coordinates are approximately $(-5.1, 1.8)$ and $(0.1, -0.488)$.

3. **Vertical Asymptotes**. These only occur when the denominator is zero, i.e., when $(x+1)(x-2) = 0$, i.e., when $x = -1, 2$.

   Horizontal/Oblique. These occur as $x \to \pm\infty$. Doing partial fractions for y, we get

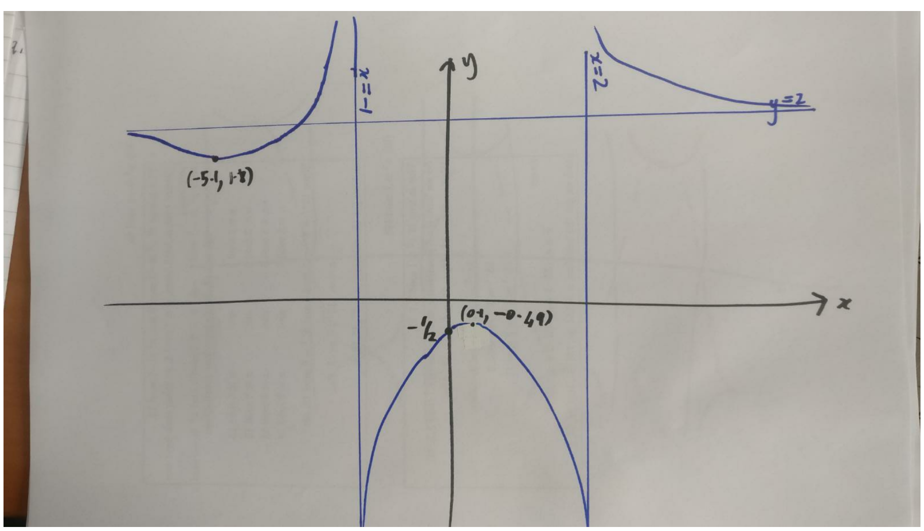
$$y = 2 + \frac{2x+5}{(x+1)(x-2)}.$$

   In general, the proper component vanishes as x is large, since $y = 2 + O(\frac{1}{x})$.

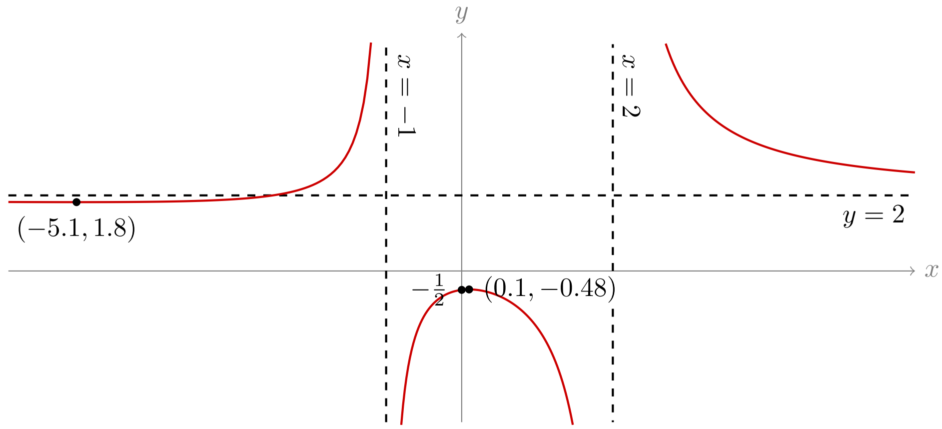
4. This last step is to help understand the behaviour at the horizontal/oblique asymptote. We can do this by taking some large values of x and substituting them in y.

x	$-\infty$	$-10\,000$	$10\,000$	∞
y	2	1.9998	2.0002	2

   So we conclude that as $x \to -\infty$, $y \to 2^-$, whereas as $x \to \infty$, $y \to 2^+$.



Here is an actual plot to compare:



LESSON 6 *13th March, 2020*

A second example:

$$y = \frac{x^2 - 5x + 6}{3x - 10}.$$

We start by finding any x- and y-intercepts.

When $x = 0$, $y = \frac{6}{-10} \implies \boxed{y = -\frac{3}{5}}$.

When $y = 0$, $x^2 - 5x + 6 = 0 \implies \boxed{x = 2, 3}$.

Now for the turning points, we set the derivative equal to zero and solve. We get

$$\begin{aligned} y' &= 0 \\ \implies (2x-5)(3x-10) - 3(x^2 - 5x + 6) &= 0 \\ \implies 3x^2 - 20x + 32 &= 0 \\ \implies (3x - 8)(x - 4) &= 0 \end{aligned}$$

So the turning points are $\boxed{(\frac{8}{3}, \frac{1}{9})}$ and $\boxed{(4, 1)}$.

Next for vertical asymptotes (poles), we check when the denominator is zero. We get that $\boxed{x = \frac{10}{3}}$ is a vertical asymptote.

For horizontal asymptotes, we approximate the behaviour of the curve when x is large in size. We do this by writing it in proper form (by long division):

$$\frac{x^2 - 5x + 6}{3x - 10} = \tfrac{1}{3}x - \tfrac{5}{9} + \frac{4/9}{3x - 10} = \tfrac{1}{3}x - \tfrac{5}{9} + O(\tfrac{1}{x}),$$

so the curve is basically the line $\frac{1}{3}x - \frac{5}{9}$ when x is large in magnitude (the notation $O(\frac{1}{x})$ denotes a quantity which is not larger than some constant times $\frac{1}{x}$. This is called big-oh notation. Don't worry about it if you're unfamiliar with it, it just allows us to focus on the important part by hiding away unnecessary details.) This means we have an oblique asymptote with equation $\boxed{9y = 3x - 5}$.

The last step is to see what happens at infinity. We can either construct a table similar to the last example:

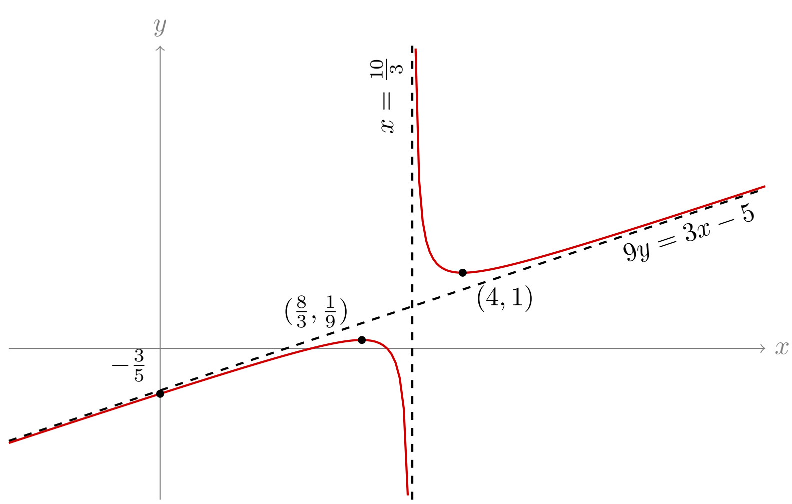
x	$-\infty$	-100	100	∞
The Asymptote ($\frac{1}{3}x - \frac{5}{9}$)	$-\infty$	-33.8889	32.7778	∞
The curve ($y = \frac{x^2-5x+6}{3x-10}$)	$-\infty$	-33.8903	32.7793	∞

From this we observe that when x is large and negative, $y < \frac{1}{3}x - \frac{5}{9}$ (i.e., the curve is below the asymptote), whereas when x is large and positive, we have that $y > \frac{1}{3}x - \frac{5}{9}$ (i.e., the curve is above the asymptote).

Perhaps something more natural we can do is again study the curve in its proper form:

$$y = \tfrac{1}{3}x - \tfrac{5}{9} + \underbrace{\frac{4/9}{3x - 10}}_{\text{diff. bet. asymptote and curve}}$$

if we study the difference when $x \to \pm\infty$, we more directly see that when x is negative and large, the denominator is negative, so the curve is below the asymptote, whereas when x is positive and large, the denominator is positive, so the curve is above the asymptote.



The Second Technique: Finding where the curve does/doesn't exist

Notice that a point on a curve

$$y = \frac{ax^2 + bx + c}{dx^2 + ex + f}$$

means that there exists a pair of numbers (x, y) such that this equation holds. In particular, if we think of y as fixed, we can ask: "is there an x such that the point (x, y) is on the curve?" This is the same as looking for solutions in x to the equation

$$\begin{aligned} y &= \frac{ax^2 + bx + c}{dx^2 + ex + f} \\ \implies (a - dy)x^2 &+ (b - ey)x + (c - fy) = 0. \end{aligned}$$

We notice that this is a quadratic, so the existence of a solution is equivalent to the fact that the discriminant Δ is non-zero, i.e.,

$$(b - ey)^2 - 4(a - dy)(c - fy) \geqslant 0.$$

What we have obtained here is a condition which y must satisfy in order for it to have a corresponding x-value. Let us see how this can help us sketch curves. For example, say we have

$$y = \frac{x^2}{1 - x}.$$

Let's find the range of values of y for which this curve has points. Rearranging as a quadratic in x, we get

$$x^2 + yx - y = 0,$$

whose discriminant is $\Delta = y^2 + 4y$, which is $\geqslant 0$ for $y \leqslant -4$ and $y \geqslant 0$. In other words, the region $-4 \leqslant y \leqslant 0$ is empty. Also, there is only one point at each of the extremities of this range, which correspond to turning points. (Reason: because these will be roots of the equation with multiplicity 2, so they survive differentiation and are turning points).

So we can find the turning points of our curve simply by setting $y = -4$ in the curve:

$$x^2 - 4x + 4 = 0 \implies x = 2 \text{ (twice)}$$

and when $y = 0$ we get $x = 0$ (twice). Thus $(2, -4)$ and $(0, 0)$ are turning points.

The rest of the method is analogous to the previous examples, where we determine intercepts, asymptotes and behaviour at infinity. This part replaces the calculus step.

5.4 The Square Root and the Modulus

TBA

LESSON 7 *26th March, 2020*

6 Further Complex Numbers

Recall that de Moivre's theorem tells how to "work out" powers of complex numbers:

$$(\cos\vartheta + i\sin\vartheta)^q = \cos q\vartheta + i\sin q\vartheta,$$

where $q \in \mathbb{Q}$. (**Exercise: prove this for integers** q **using induction.**) Writing this differently, we have

$$(e^{i\vartheta})^q = e^{iq\vartheta}$$

The other facts about powers are not necessarily true, in particular, $(xy)^a \neq x^a y^a$ in general. The important ones which remain true are $x^a x^b = x^{a+b}$ and $(x^a)^b = x^{ab}$.

6.1 Obtaining Trigonometric Identities

Recall that, e.g., $\sin 2x = 2\sin x\cos x$. We also had triple angle identities, $\sin 3x = 3\sin x - 4\sin^3 x$, and so on. To obtain these, we used the compound angle identity repeatedly, (i.e., first do $\sin(x+x)$ for identities on $\sin 2x$, and then $\sin(2x+x)$ for $\sin 3x$, and so on).

There is an easier way, using de Moivre's theorem. Let's obtain an identity for $\cos 4x$.

$$\begin{aligned}
\cos 4x &= \Re(\cos 4x + i\sin 4x)\\
&= \Re((\cos x + i\sin x)^4)\\
&= \Re((c+is)^4)\\
&= \Re(c^4 + 4c^3 is + 6c^2 i^2 s^2 + 4ci^3 s^3 + i^4 s^4)\\
&= \Re(c^4 - 6c^2 s^2 + s^4 + i(4c^3 s - 4cs^3))\\
&= c^4 - 6c^2 s^2 + s^4.
\end{aligned}$$

In other words,

$$\cos 4x = \cos^4 x - 6\cos^2 x\sin^2 x + \sin^4 x.$$

We also immediately get an identity for $\sin 4x$:

$$\sin 4x = \Im(c^4 - 6c^2 s^2 + s^4 + i(4c^3 s - 4cs^3)) = 4\cos x\sin x(\cos^2 x - \sin^2 x).$$

If we insist of only having cosines for the first identity, we can make use of Pythagoras:

$$\begin{aligned}\cos 4x &= \cos^4 x - 6\cos^2 x \sin^2 x + \sin^4 x \\ &= \cos^4 x - 6\cos^2 x(1-\cos^2 x) + (1-\cos^2 x)^2 \\ &= \cos^4 x - 6\cos^2 x + 6\cos^4 x + 1 - 2\cos^2 x + \cos^4 x \\ &= 8\cos^4 x - 8\cos^2 x + 1.\end{aligned}$$

We can also use these identities to obtain an identity for the tangent function.

$$\begin{aligned}\tan 4x &= \frac{\sin 4x}{\cos 4x} \\ &= \frac{4\cos^3 x \sin x - 4\cos x \sin^3 x}{\cos^4 x - 6\cos^2 x \sin^2 x + \sin^4 x} \qquad (\div \cos^4 x) \\ &= \frac{4\frac{\sin x}{\cos x} - 4\frac{\sin^3 x}{\cos^3 x}}{1 - 6\frac{\sin^2 x}{\cos^2 x} + \frac{\sin^4 x}{\cos^4 x}} \\ &= \frac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}.\end{aligned}$$

Now perhaps the more useful case for A-level will be the other way around, i.e., expressing powers of sines and cosines in terms of multiple angles (for integration and so on).

Let $z = \cos x + i\sin x$. Then by DMT, $z^n = \cos nx + i\sin nx$ and $z^{-n} = \cos nx - i\sin nx$. In particular, we get that

$$\boxed{z^n + z^{-n} = 2\cos nx} \qquad \text{and} \qquad \boxed{z^n - z^{-n} = 2i\sin nx}.$$

Let's say we want an identity for $\cos^4 2x$ (in terms of multiple angles only, no powers). Put $n = 2$ in the cosine formula:

$$\begin{aligned}z^2 + z^{-2} &= 2\cos 2x \\ \implies (z^2 + z^{-2})^4 &= 2^4 \cos^4 2x \\ \implies z^8 + 4z^4 + 6 + 4z^{-4} + z^{-8} &= 2^4 \cos^4 2x \\ \implies z^8 + z^{-8} + 4(z^4 + z^{-4}) + 6 &= 2^4 \cos^4 2x \\ \implies 2\cos 8x + 8\cos 4x + 6 &= 2^4 \cos^4 2x\end{aligned}$$

and therefore the identity is

$$\cos^4 x = \tfrac{1}{8}(\cos 8x + 4\cos 4x + 3).$$

6.2 Some Neat Examples

The complex exponential has certain structure which we can take advantage of, which the trigonometric functions do not. For instance, consider the following sum.

$$\sum_{k=1}^{n} \cos(kx).$$

Say we want to determine a closed form for this sum. Notice that $\cos kx = \Re(\cos kx + i\sin kx) = \Re e^{ikx} = \Re(e^{ix})^k$. So what we have is actually

$$\sum_{k=1}^{n} \Re(e^{ix})^k = \Re\sum_{k=1}^{n}(e^{ix})^k.$$

We can recognise this as a geometric series:

$$\sum_{k=1}^{n}(e^{ix})^k = e^{ix}\,\frac{1-(e^{ix})^n}{1-e^{ix}} = \frac{e^{ix}-(e^{ix})^{n+1}}{1-e^{ix}}$$

Now it is "good to know" the following identities:

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \qquad \text{and} \qquad \sin x = \frac{e^{ix}-e^{-ix}}{2i},$$

these indicate a way forward to simplify the denominator. We would like the denominator to resemble these cosine/sine identities:

$$\frac{e^{ix}-(e^{ix})^{n+1}}{1-e^{ix}} = \frac{e^{-ix}(e^{ix}-(e^{ix})^{n+1})}{e^{-ix/2}-e^{ix/2}} = \frac{e^{-ix/2}(e^{ix}-(e^{ix})^{n+1})}{2i\sin\frac{x}{2}}$$

Now we deal with the numerator. We have

$$\frac{-i}{2\sin\frac{x}{2}}\big(e^{ix/2}-(e^{ix})^{n+1-\frac{1}{2}}\big)$$

which equals

$$\frac{-i}{2\sin\frac{x}{2}}\big(\cos\tfrac{x}{2}+i\sin\tfrac{x}{2}-\cos((n+\tfrac{1}{2})x)+i\sin((n+\tfrac{1}{2})x)\big),$$

and distributing out the i,

$$-\tfrac{1}{2}\csc\tfrac{x}{2}(i\cos\tfrac{x}{2}-\sin\tfrac{x}{2}-i\cos((n+\tfrac{1}{2})x)-\sin((n+\tfrac{1}{2})x)),$$

whose real part is

$$\tfrac{1}{2}\csc\tfrac{x}{2}(\sin\tfrac{x}{2}+\sin((n+\tfrac{1}{2})x)).$$

We can simplify this further using $\sin A+\sin B = 2\sin((A+B)/2)\cos((A+B)/2)$ to get the final answer

$$\sum_{k=1}^{n}\cos(kx) = \csc\tfrac{x}{2}\sin\tfrac{nx}{2}\cos(\tfrac{1}{2}(n+1)x).$$

We can similarly obtain a sum for the sine (by taking the imaginary part instead). What about something like

$$\sum_{k=1}^{n} k\cos(kx)?$$

We can do the following neat trick. Notice that we can get a k in front of the general term z^k in

$$\sum_{k=1}^{n} z^k$$

by differentiating:

$$\frac{d}{dz}\sum_{k=1}^{n} z^k = \sum_{k=1}^{n}\frac{d}{dz}z^k = \sum_{k=1}^{n} kz^{k-1},$$

and multiplying throughout by z:

$$z\,\frac{d}{dz}\sum_{k=1}^{n} z^k = \sum_{k=1}^{n} k\,z^k.$$

But we know what the LHS is:

$$z\,\frac{d}{dz}\Big(z\,\frac{1-z^n}{1-z}\Big) = \sum_{k=1}^{n} k\,z^k.$$

which we can simplify to get that

$$\sum_{k=1}^{n} k\,z^k = \frac{(n(z-1)-1)z^{n+1}+z}{(z-1)^2},$$

and then if we put $z = e^{ix}$ and take the real part, the LHS becomes the sum we want, and we can simplify the RHS in a similar way to the last example, to get the following final result: https://math.stackexchange.com/a/3611630/301095.

Question 8, problems sheet. We have that two of the roots are $\zeta = 1-\sqrt{3}i$ and $\bar{\zeta} = 1+\sqrt{3}i$. So the polynomial $f = z^4+6z^3+13z^2-18z+100$ has $(z-\zeta)(z-\bar{\zeta}) = z^2-2z+4$ as a factor. Thus we have

$$f = z^4+6z^3+13z^2-18z+100 = (z^2-2z+4)(z^2+Az+25)$$

where A is a constant to be determined. If we expand the rhs, the coefficient of z^3 will be $-2+A$, and this should equal 6, so $A = 8$. Thus

$$f = (z^2 - 2z + 4)(z^2 + 8z + 25)$$

Thus the remaining two roots of f are the roots of $z^2 + 8z + 25$, which we can find by completing the square to be $-4 \pm 3i$.

Question 11b, problem sheet.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} e^\theta \cos 2\theta \, d\theta &= \int_0^{\frac{\pi}{2}} e^\theta \Re e^{2i\theta} \, d\theta \\
&= \Re \int_0^{\frac{\pi}{2}} e^{(1+2i)\theta} \, d\theta \\
&= \Re \Big(\frac{e^{(1+2i)\theta}}{1+2i} \Big)_0^{\frac{\pi}{2}} \\
&= \Re \Big(\frac{1-2i}{5} \, e^{(1+2i)\theta} \Big|_0^{\frac{\pi}{2}} \Big) \\
&= \Re \Big(\frac{1-2i}{5} (e^{(1+2i)\frac{\pi}{2}} - 1) \Big) \\
&= \Re \Big(\frac{1-2i}{5} (e^{\frac{\pi}{2}} e^{i\pi} - 1) \Big) \\
&= \Re \Big(\frac{1-2i}{5} (-e^{\frac{\pi}{2}} - 1) \Big) = -\frac{e^{\frac{\pi}{2}} - 1}{5}
\end{aligned}$$

6.3 Complex Loci

We can consider subsets of the complex plane defined by equations, just as we do in Coordinate geometry and polar coordinates. For instance,

$$|z| = 3$$

represents the equation of a circle centred at the origin, with radius 3 (since it's all complex numbers of modulus 3). An easy way to double check what a complex locus is, is to substitute $z = x + iy$, this will translate the equation into a Cartesian one about points of $\mathbb{R}^2$. The above becomes

$$|x + iy| = 3 \iff \sqrt{x^2 + y^2} = 3 \iff x^2 + y^2 = 9$$

We have three "main" complex loci we should recognise:

- $|z - \zeta| = r$
- $|z - \zeta| = |z - \omega|$
- $\arg(z - \zeta) = \vartheta$

The first one is a circle, centred at ζ with radius r. Indeed, if $\zeta = a + bi$, then putting $z = x + iy$, we have

$$|(x+iy)-(a+bi)| = r \iff \sqrt{(x-a)^2 + (y-b)^2} = r \iff (x-a)^2+(y-b)^2 = r^2.$$

The second locus represents a straight line which is the perpendicular bisector of the line segment joining ζ to ω. (A good exercise to put $z = x + iy$, etc, to get the linear equation).

The last one represents a part-line starting from ζ in the direction of the angle θ.