I. INTRODUCTION

THE *unit circle* is the circle centred at the origin with radius 1. Formally, it is the subset $\mathscr{C} \subseteq \mathbb{R}^2$ of points $(x, y) \in \mathbb{R}^2$ satisfying the equation $x^2 + y^2 = 1$.

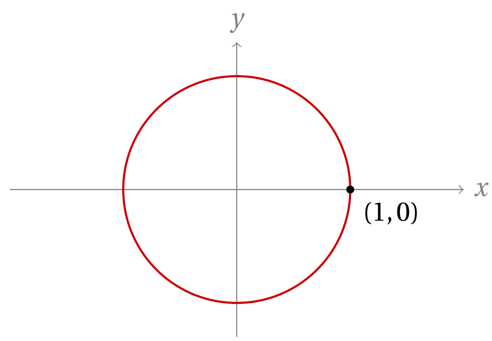

FIGURE 1: The unit circle $\mathscr{C} : x^2 + y^2 = 1$.

An *angle* $\vartheta \in \mathbb{R}$ is simply a real number, which in the context of trigonometry, we interpret as a *distance* travelled anticlockwise along the unit circle, starting from the point $(1,0)$. The notion of "curved distance" requires calculus to formalise properly. Let us briefly discuss how this can be done, without getting into the heavy details. Suppose for now that $0 \leqslant \vartheta \leqslant 2$.

The construction of a line segment of length ϑ such that one end of the segment is at $(1,0)$, and the other is at a point on the circle above the x-axis, uniquely determines a point P_1 (see figure 2).

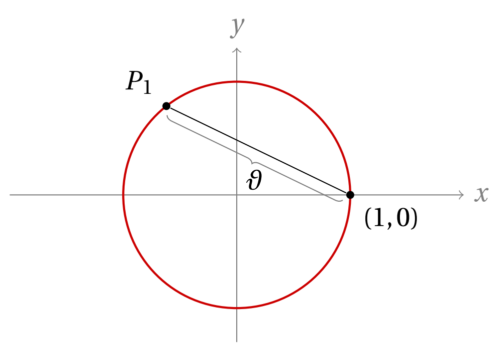

FIGURE 2: The point P_1, uniquely determined by the line segment of length ϑ.

In fact, it is not hard to show that

$$P_1 = \left(1 - \frac{\vartheta^2}{2}, \frac{\vartheta}{n}\sqrt{1 - \frac{\vartheta^2}{4}}\right).$$

Next, if we divide this line segment into two segments of length ${}^{\vartheta}/_{2}$, joining them tail to tip at another point of the circle, this similarly determines a point P_2 (see figure 3). Notice the combined length is still ${}^{\vartheta}/_{2} + {}^{\vartheta}/_{2} = \vartheta$.

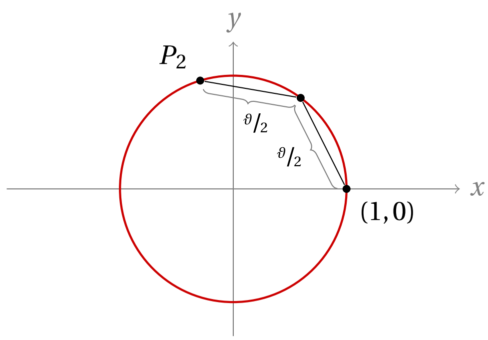

FIGURE 3: The point P_2, uniquely determined by two touching line segments of length ${}^{\vartheta}/_{2}$.

We can similarly divide the line segment into three pieces of length ${}^{\vartheta}/_{3}$, determining a point P_3, retaining combined length ${}^{\vartheta}/_{3} + {}^{\vartheta}/_{3} + {}^{\vartheta}/_{3} = \vartheta$. Continuing this

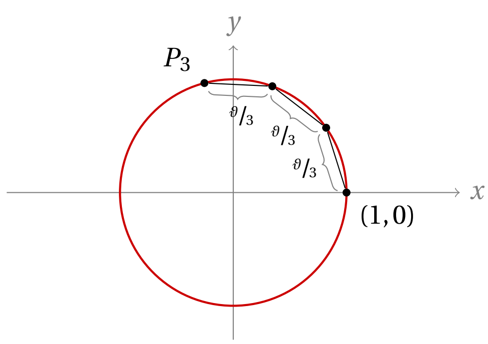

FIGURE 4: The point P_3, uniquely determined by three touching line segments of length ${}^{\vartheta}/_{3}$.

way, we can for any n, determine a point P_n, obtained by joining n line segments of length ${}^{\vartheta}/_{n}$, tail to tip, at other points on the circle. In each case, the combined length of the segments is always ϑ. It can be shown (by induction, say) that the x-coordinate x_n of P_n is given by

$$x_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} \left(1 - \frac{(\vartheta/n)^2}{2}\right)^{n-2k} \left(\frac{\vartheta}{n}\right)^{2k} \left(1 - \frac{(\vartheta/n)^2}{4}\right)^{k}, \tag{1}$$

and similarly, the y-coordinate y_n is given by

$$y_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k+1} \left(1 - \frac{(\vartheta/n)^2}{2}\right)^{n-2k-1} \left(\frac{\vartheta}{n}\right)^{2k+1} \left(1 - \frac{(\vartheta/n)^2}{4}\right)^{k+1/2}. \tag{2}$$

In the limit of this process (in the calculus sense), we end up with our desired “curved” distance travelled, at a point we shall call P_∞.

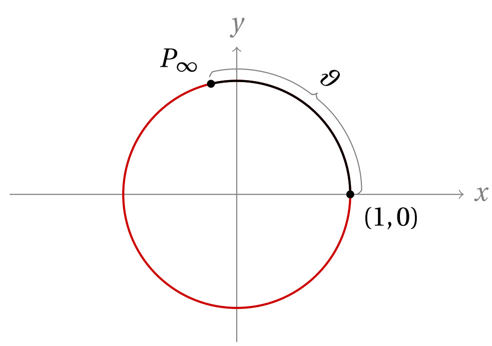

FIGURE 5: The point P_∞, at a curved distance of ϑ from $(1,0)$.

Now if $\vartheta > 2$, the initial line segment P_1 will exit the circle, so instead we start at a later stage, namely, with some P_n such that $\vartheta/n \leqslant 2$. Furthermore, if $\vartheta < 0$, then we instead travel a distance of $-\vartheta$ in the clockwise direction. In both cases, it can be shown that the formulæ in (1) and (2) remain valid. Thus we can determine P_∞ for any angle $\vartheta \in \mathbb{R}$.

DEFINING THE CIRCULAR FUNCTIONS

Definition 1 (Sine and cosine)**.** Let $\vartheta \in \mathbb{R}$ be an angle, and perform the described process to obtain $P_\infty = (x, y)$. Since these two values depend solely on ϑ, we define the *cosine function*, denoted by $\cos\vartheta$, to be the x-coordinate of P_∞, and the *sine function*, denoted by $\sin\vartheta$, to be the y-coordinate at P_∞.

In other words, we have $P_\infty = (\cos\vartheta, \sin\vartheta)$. These functions are called *circular* or *trigonometric*.

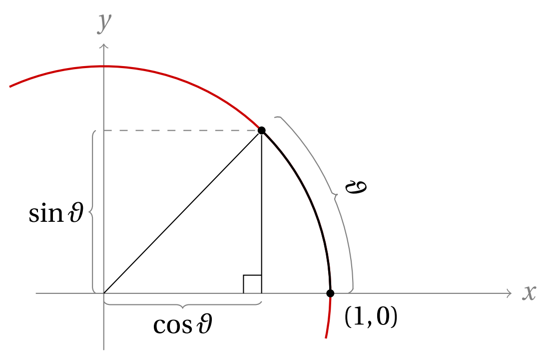

FIGURE 6: The functions $\cos\vartheta$ and $\sin\vartheta$.

Remark 2 (Maclaurin series). If you have not seen power series yet, you can safely ignore this remark. As $n\to\infty$, one can show that

$$\binom{n}{2k}\frac{1}{n^{2k}}\to\frac{1}{(2k)!},\qquad \left(1-\frac{(\vartheta/n)^2}{2}\right)^{n-2k}\to 1\qquad\text{and}\qquad \left(1-\frac{(\vartheta/n)^2}{4}\right)^{k}\to 1.$$

It follows that the kth term in (1) becomes

$$\frac{(-1)^k}{(2k)!}\,\vartheta^{2k},$$

and so, glossing over some details of convergence (swapping the limit and the sum), we have that

$$\cos\vartheta=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\,\vartheta^{2k},$$

which is usually given as the definition of the cosine function in more formal treatments. Identically, from (2), we obtain that

$$\sin\vartheta=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\,\vartheta^{2k+1}.$$

Notation. The squares $(\cos\vartheta)^2$ and $(\sin\vartheta)^2$ of the trigonometric functions are denoted by $\cos^2\vartheta$ and $\sin^2\vartheta$. More generally, for $n\geqslant 2$, $f^n(x)$ denotes $(f(x))^n$.

What links the sine and cosine functions is the fact that they represent a point on the unit circle $\mathscr{C}$. This is summarised in the following theorem.

Theorem 3 (The Pythagorean Identity)**.** *Let $\vartheta\in\mathbb{R}$. Then*

$$\cos^2\vartheta+\sin^2\vartheta=1.$$

Proof. By definition, the point $(\cos\vartheta, \sin\vartheta)$ lies on the unit circle, so satisfies the equation $x^2 + y^2 = 1$. □

Another important property about the sine and cosine is that they are bounded in size by 1. This is intuitive, since they are coordinates of points on the circle.

Theorem 4. *Let* $\vartheta \in \mathbb{R}$. *Then*

$$-1 \leqslant \cos\vartheta \leqslant 1 \qquad \textit{and} \qquad -1 \leqslant \sin\vartheta \leqslant 1.$$

Proof. We have $|\cos\vartheta| = \sqrt{1-\sin^2\vartheta} \leqslant \sqrt{1-0} = 1$ since $\sin^2\vartheta \geqslant 0$ and the square root function is increasing, and similarly $|\sin\vartheta| = \sqrt{1-\cos^2\vartheta} \leqslant 1$. □

The sine and cosine are the most important trigonometric functions, but there are others. Each function corresponds to some length when we look at the picture of ϑ on the unit circle, which can be seen in figure 7.

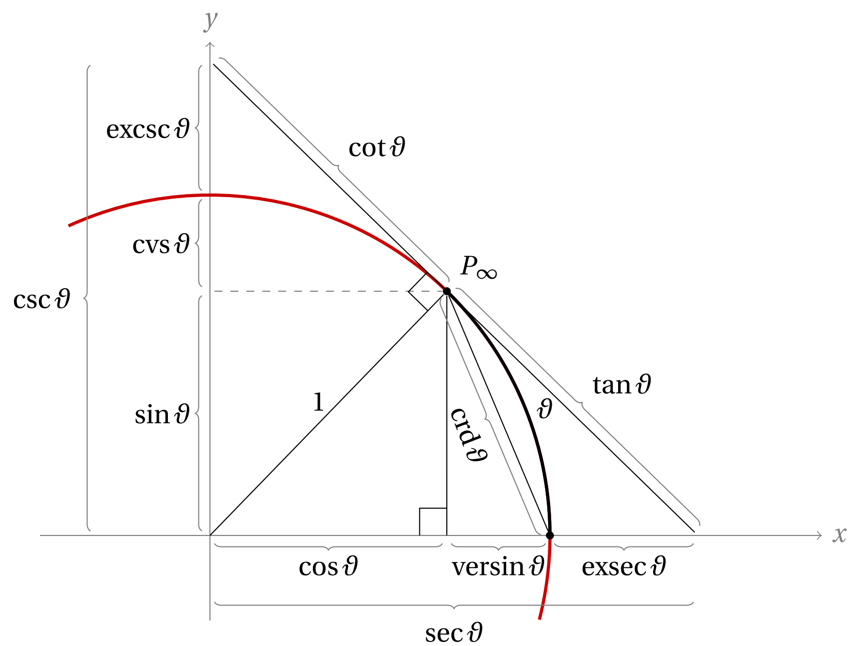

FIGURE 7: The trigonometric functions

The reason these are "less" important is that we can easily express them in terms of sine and cosine, rendering them redundant, in a sense. Usually, the main six functions are grouped as follows.

- **Major Trigonometric Functions.** This group consists of the sine, cosine, and *tangent* functions, the latter of which is denoted by $\tan\vartheta$. This function is the length of the tangent to the circle at the point P_∞, from P_∞ to its x-intercept. It turns out that

$$\tan\vartheta = \frac{\sin\vartheta}{\cos\vartheta}.$$

- **Minor Trigonometric Functions.** This group consists of the *secant*, *cosecant* and *cotangent* functions, denoted $\sec\vartheta$, $\csc\vartheta$ (sometimes $\operatorname{cosec}\vartheta$), and $\cot\vartheta$ respectively. The secant and cosecant measure the x- and y-intercepts of the tangent line at P_∞, and the cotangent measures the length from P_∞ to the tangent's y-intercept. It turns out that

$$\sec\vartheta = \frac{1}{\cos\vartheta}, \qquad \csc\vartheta = \frac{1}{\sin\vartheta}, \quad \text{and} \quad \cot\vartheta = \frac{\cos\vartheta}{\sin\vartheta} = \frac{1}{\tan\vartheta}.$$

The remaining functions are the *versine* (versed sine), *coversine*, *exsecant* (exterior secant), *excosecant* and the *chord*, denoted by $\operatorname{versin}\vartheta$ (sometimes $\operatorname{vrs}\vartheta$), $\operatorname{cvs}\vartheta$, $\operatorname{exsec}\vartheta$ (sometimes $\operatorname{exs}\vartheta$), $\operatorname{excsc}\vartheta$ (sometimes $\operatorname{exc}\vartheta$) and $\operatorname{crd}\vartheta$, respectively. The easiest way to explain what each of these are is to direct the reader to figure 7. The chord is the length of the chord from $(1,0)$ to the point P_∞. These functions have been popular historically, but are seldom used today, so we will not be using them. We only mention them here for completeness. In terms of the major and minor trigonometric functions, we have:

$$\operatorname{versin}\vartheta = 1-\cos\vartheta \qquad \operatorname{cvs}\vartheta = 1-\sin\vartheta$$
$$\operatorname{exsec}\vartheta = \sec\vartheta - 1 = \frac{1-\cos\vartheta}{\cos\vartheta} \qquad \operatorname{excsc}\vartheta = \csc\vartheta - 1 = \frac{1-\sin\vartheta}{\sin\vartheta}$$
$$\operatorname{crd}\vartheta = 2\sin{}^{\vartheta}\!/_{2}.$$

Observe that the "co-" prefix to each trigonometric function respects the symmetry of the diagonal in figure 7. Notice also that Pythagoras' theorem applied to the two right-angled triangles with hypotenuse $\sec\vartheta$ and $\csc\vartheta$ respectively gives us the following.

Corollary 5 (Pythagorean Identities)**.** *Let* $\vartheta \in \mathbb{R}$. *Then*

$$1+\tan^2\vartheta = \sec^2\vartheta \qquad \textit{and} \qquad 1+\cot^2\vartheta = \csc^2\vartheta,$$

where the first holds if $\cos\vartheta \neq 0$*, and the second holds if* $\sin\vartheta \neq 0$.

Proof. Divide the identity $\cos^2\vartheta + \sin^2\vartheta = 1$ of theorem 3 by $\cos^2\vartheta$ for the first identity, and by $\sin^2\vartheta$ for the second. □

II. GRAPHS AND PROPERTIES OF THE CIRCULAR FUNCTIONS

IN this section, we will aim to plot the major trigonometric functions as they vary with ϑ. In order to reason about their graphs, we must first define some nice properties which real-valued functions might have, and develop some theory around them.

ODD AND EVEN FUNCTIONS

Definition 6 (Even and odd functions)**.** Let $A \subseteq \mathbb{R}$. A function $f\colon A \to \mathbb{R}$ is said to be *even* if for all $x \in A$,

$$f(-x) = f(x),$$

whereas it is said to be *odd* if for all $x \in A$,

$$f(-x) = -f(x).$$

For example, $f(x) = x^2$ is an even function, and $g(x) = x^3$ is an odd function. The function $h(x) = x + 1$ is neither even nor odd. Graphically, an even function must be symmetric in the y-axis, since (x, y) is a point on $y = f(x)$ if and only if $(-x, y)$ is. On the other hand, an odd function must have rotational symmetry about the origin, since (x, y) is a point on $y = f(x)$ if and only if $(-x, -y)$ is. Refer to figure 8.

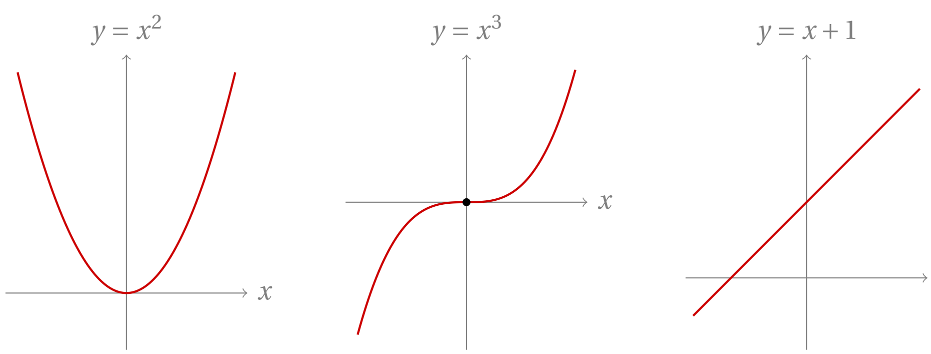

FIGURE 8: Plots of $y = x^2$, $y = x^3$ and $y = x + 1$. Notice that $y = x^2$ is symmetric in the y-axis, $y = x^3$ has rotational symmetry about the origin, and $y = x + 1$ has neither of these properties.

Proposition 7. *Let $A \subseteq \mathbb{R}$, let $f_e, g_e\colon A \to \mathbb{R}$ be a pair of even functions, and let $f_o, g_o\colon A \to \mathbb{R}$ be a pair of odd functions. Then*

(i) $f_e \pm g_e$ is even,
(ii) $f_o \pm g_o$ is odd,
(iii) $f_e g_e$ is even,
(iv) $f_o g_o$ is even.

Proof. We simply use the definitions of even and odd. For (i),

$$(f_e \pm g_e)(-x) = f_e(-x) \pm g_e(-x) = f_e(x) \pm g_e(x) = (f_e \pm g_e)(x),$$

and for (ii),

$$\begin{aligned}(f_o \pm g_o)(-x) = f_o(-x) \pm g_o(-x) &= -f_o(x) \mp g_o(x)\\ &= -(f_o(x) \pm g_o(x))\\ &= -(f_o \pm g_o)(x),\end{aligned}$$

the proofs of (iii) and (iv) are similar. □

PERIODIC FUNCTIONS

Definition 8 (Periodic function)**.** Let $A \subseteq \mathbb{R}$. A function $f\colon A \to \mathbb{R}$ is said to be *periodic* (with period T) if there exists $T > 0$ such that

$$f(x+T) = f(x)$$

for all $x \in A$. If there exists a number $T_0 > 0$ such that f is periodic with period T_0, and for all $T \in (0, T_0)$, f is not periodic with period T, then T_0 is said to be the *fundamental period* of f. (In other words, the fundamental period is the smallest possible T.)

Graphically, a periodic function exhibits translational symmetry.

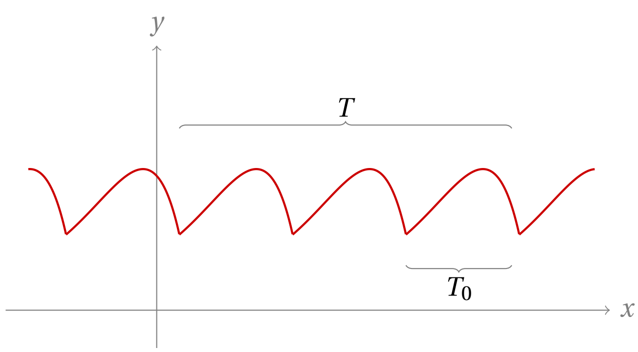

FIGURE 9: The graph of a periodic function. T is a possible period, T_0 is the fundamental period.

Proposition 9. *Let $f\colon A \to \mathbb{R}$ be periodic with fundamental period T_0. Then each period of f is an integer multiple of T_0, and $f(x + nT_0) = f(x)$ for all $n \in \mathbb{Z}$.*

Proof. We start with the second part. If $n \geqslant 0$, applying the periodicity of f n times, we have

$$\begin{aligned} f(x+nT_0) &= f(x+\underbrace{T_0+\cdots+T_0}_{n \text{ times}}) \\ &= f(x+\underbrace{T_0+\cdots+T_0}_{n-1 \text{ times}}) \\ &\vdots \\ &= f(x+T_0) \\ &= f(x) \end{aligned}$$

for all x. On the other hand, if $n < 0$, then again applying periodicity $-n$ times, we get

$$f(x+nT_0) = f(x+nT_0+T_0) = \cdots = f(x+nT_0+(-n)T_0) = f(x),$$

for all x, as required.

Now for the first part, suppose f is periodic with period T, but T is not an integer multiple of T_0. Notice

$$\mathbb{R} = \cdots [-2T_0, -T_0] \cup [-T_0, 0] \cup [0, T_0] \cup [T_0, 2T_0] \cup \cdots,$$

and since T is not one of the end points, it lies precisely in one of the intervals $[nT_0, (n+1)T_0]$ for some $n \in \mathbb{Z}$. So we have

$$\begin{aligned} &nT_0 < T < nT_0 + T_0 \\ \implies &0 < T - nT_0 < T_0. \end{aligned}$$

Let $T' = T - nT_0$. Then for all x, $f(x+T') = f((x+T) - nT_0) = f(x+T)$ by the second part of the proposition, and by periodicity, this equals $f(x)$. In other words, f is periodic with period T'. But $T' < T_0$, and T_0 is a the fundamental period! This is a contradiction, so there cannot be a period which is not an integer multiple of T_0. □

To state the next proposition, we need some notation of sets.

Notation (Set operations)**.** Let $A, B \subseteq \mathbb{R}$, and $x \in \mathbb{R}$. Then we adopt the following notations.

- $A + B = \{a + b : a \in A \text{ and } b \in B\}$,
- $A - B = \{a - b : a \in A \text{ and } b \in B\}$,

- $x + A = \{x + a : a \in A\}$ and $A + x = \{a + x : a \in A\}$,
- $xA = \{xa : a \in A\}$ and $Ax = \{ax : a \in A\}$.

Be careful with this notation, although the definitions mirror closely the corresponding operations on numbers, not all properties follow, e.g., $2A \neq A + A$.

The next proposition tells us about solving equations involving periodic functions.

Proposition 10. *Let $f\colon A \to \mathbb{R}$ be a periodic function with fundamental period T_0, let $\alpha \in \mathbb{R}$, and let X be the set of solutions of the equation $f(x) = \alpha$ in the range $[a, a + T_0)$ for some $a \in \mathbb{R}$. Then the set of solutions of $f(x) = \alpha$ over A is*

$$(X + T_0\mathbb{Z}) \cap A.$$

Proof. Let S be the set of all solutions of $f(x) = \alpha$ over A. We want to show that $S = (X + T_0\mathbb{Z}) \cap A$. We will do this by showing that each is a subset of the other. First, take $x \in (X + T_0\mathbb{Z}) \cap A$. Then $x = x' + nT_0$ for some $x' \in X$ and $n \in \mathbb{Z}$. Thus

$$f(x) = f(x' + nT_0) = f(x') = \alpha$$

by proposition 9, and so $x \in S$. It follows that $(X + T_0\mathbb{Z}) \cap A \subseteq S$.

Next, take $x \in S$. Since

$$\mathbb{R} = \cdots [a - 2T_0, a - T_0) \cup [a - T_0, a) \cup [a, a + T_0) \cup [a + T_0, a + 2T_0) \cup \cdots,$$

Then x lies in precisely one of the intervals $[a + nT_0, a + (n+1)T_0]$ for some $n \in \mathbb{Z}$. So we have

$$\begin{aligned} & a + nT_0 < x < a + nT_0 + T_0 \\ \Longrightarrow\ & a < x - nT_0 < a + T_0. \end{aligned}$$

Set $x' = x - nT_0$. Then $x = x' + nT_0 \in X + T_0\mathbb{Z}$, and since $x \in S \subseteq A$, it follows that $x \in (X + T_0\mathbb{Z}) \cap A$, so $S \subseteq (X + T_0\mathbb{Z}) \cap A$. □

ANGLE FACTS

Now let us think a bit about angles. How large does ϑ have to be so that we traverse a semicircle, i.e., what is the smallest $\vartheta > 0$ such that $\cos\vartheta = -1$?

Definition 11 (π)**.** The smallest positive real number such that

$$\cos\vartheta = -1$$

is denoted by π.