CHAPTER 2

SEQUENCES & SUMS

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1 Sequences

A *sequence* or *progression* is an ordered list of real numbers, which we call the *terms* of the sequence. We usually denote the terms by

$$a_1, a_2, \ldots, a_n, \ldots,$$

where a_1 denotes the first term, a_2 denotes the second, and so on; a_n in general denotes the "nth" term.

Examples 1.1. (i) The positive odd numbers

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots,$$

are an example of a sequence. The first term is $a_1 = 1$, the fifth term is $a_5 = 9$. We can write the nth term in general as $a_n = 2n - 1$. For instance, $a_{20} = 2 \cdot 20 - 1 = 39$.

(ii) A (boring) example of a sequence is

$$\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots,$$

whose terms area all $\frac{1}{2}$. This has nth term $b_n = \frac{1}{2}$ for all n.

(iii) The Fibonacci sequence is a famous sequence constructed as follows. Starting with 1 and 1, each next term is the sum of the previous two. It goes

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

If we denote the nth term by F_n, we have $F_1 = 1$ and $F_{10} = 55$ (for example). Our definition can be written as a formula for the nth term in general, i.e.,

$$F_n = \begin{cases} 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ F_{n-1} + F_{n-2} & \text{otherwise,} \end{cases}$$

although this definition is what's called *recursive*, i.e., we can't simply plug in, say, $n = 50$, and obtain F_{50}, without having already worked out F_{49} and F_{48} (which in turn require us to calculate F_{48} and F_{47}, and so on, until we go all the way down to F_1 and F_2).

It turns out that there is an explicit formula for F_n which does not require us to perform as many computations, and it's given by

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

This formula is very interesting—particularly because of the appearance of $\sqrt{5}$, despite the fact that for every n, it is a whole number. We will not delve deeper into how this formula is obtained, but it should give you the impression that very interesting theory can arise from simply defined sequences.

(iv) The primes are one of the most enticing sequences for mathematicians: these are the whole numbers which have no divisors apart from 1 and themselves:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \ldots$$

There is no known formula for the nth prime, however one of the most famous results in mathematics, the *prime number theorem* (PNT), tells us that it is around $n \log n$.[1] Moreover, we have that for all $n \geqslant 6$, the nth prime p_n satisfies the inequality

$$n \log n + n \log(\log n) - n < p_n < n \log n + n \log(\log n).$$

E.g., when $n = 10\,000$, this becomes $104\,306.67 < p_{10\,000} < 114\,306.67$; the actual value is $p_{10\,000} = 104\,729$. Understanding better how this sequence behaves is tied to one of the most famous unsolved problems in all of mathematics: the Riemann hypothesis.

[1] $\log(n)$ is an important function we will discuss more later, the only important thing to note here is that it grows very slowly, much slower than n (for instance, $\log(1000) \approx 6.9$).

Exercise 1.2. 1. A sequence has its nth term given by $a_n = 2n-3$ for all n. Write down the first five terms of this sequence.

2. A sequence is given by $b_n = n^2 + 1$ for all n. Write down the first five terms of this sequence. What is the tenth term?

3. The *Lucas sequence* is a variant of the Fibonacci sequence governed by the same rule, i.e., $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 3$, but it starts with 2, 1 instead of 1, 1. Write down the first ten terms of the Lucas sequence.

4. Write down the first five terms of the sequence defined by the formula $c_n = (-1)^{n+1}/\sqrt{n}$.

Our goal in the MATSEC intermediate is to understand only the most basic kinds of sequences. Possibly the most natural sequences we can consider as a starting point are:

1.1 Arithmetic Sequences

The word "arithmetic" is usually synonymous with "additive", i.e., it is an adjective describing something involving addition. Indeed, an *arithmetic progression* (AP) is what we call a sequence where we always add the same number to get from one term to the next. For instance,

$$7, 11, 15, 19, 23, 27, 31, 35, 39, 43, \ldots$$

is an arithmetic sequence, since we always add 4 to get from one term to the next one. Similarly, the sequence of odd numbers is an arithmetic progression (any two successive odd numbers differ by 2), so is

$$23, 20, 17, 14, 11, 8, 5, 2, -1, -4, \ldots$$

since we are adding -3 each time. In general, if a denotes the first term of an AP, and we are adding d each time, then the sequence must be

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad a+4d, \quad a+5d, \quad \ldots$$

The number d is referred to as the *common difference*, since it is the difference between any two successive terms.

Notice the slightly annoying fact that the coefficient of d in a term is offset from the term's position in the sequence by 1:

$$\underbrace{a+0d}_{1\text{st term}}, \quad \underbrace{a+1d}_{2\text{nd term}}, \quad \underbrace{a+2d}_{3\text{rd term}}, \quad \underbrace{a+3d}_{4\text{th term}}, \quad \ldots, \quad \underbrace{a+(n-1)d}_{n\text{th term}}, \quad \ldots$$

Indeed, it would be nice if an AP which has first term 7 and has common difference of 4 (say) would correspond to the nth term $a_n = 7 + 4n$, but if we plug $n = 1$ into this expression we would actually end up with the second term 11 instead of the first term. We fix this by offsetting n by 1, so the nth term is actually $a_n = 7 + 4(n - 1) = 3 + 4n$.

Thus, if an AP has nth term $b + cn$, then b is *not* the first term, its first term is $b + 1n$ (if we like, we could call b the *zeroth* term of the sequence). On the other hand, we can just read off the common difference, since it is the coefficient c of n.

Example 1.3. If 2 and 14 are the first and fifth terms of an AP respectively, what is the seventh term? What is the nth term?

We can rewrite the given information in terms of equations about a and d. Indeed, we have

$$\begin{cases} a \quad\quad = 2 & \text{①} \\ a + 4d = 14 & \text{②} \end{cases}$$

Subtracting ② − ① gives $4d = 12 \implies d = 3$, and so the seventh term is $a + 6d = 2 + 18 = 20$, and the nth term in general is $a_n = a + (n - 1)d = 2 + 3(n - 1) = 3n - 1$. □

Example 1.4. The sum of the first five terms of an AP is equal to ten times the seventh term, and the sum of the third and fourth terms is 10. What is the nth term?

This is a seemingly more complicated problem, but again, the key is to simply translate the information given into equations. The first sentence is telling us that

$$a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) = 10 \times (a + 6d),$$

which simplifies to $a = -10d$. The second bit of information tells us that

$$(a + 2d) + (a + 3d) = 10,$$

which simplifies to $2a + 5d = 10$. Substituting the first equation into the second gives us

$$2(-10d) + 5d = 10 \implies -15d = 10 \implies d = -\tfrac{2}{3}.$$

Thus the first term is $a = -10d = \frac{20}{3}$, and so the nth term is

$$a_n = a + (n - 1)d = \tfrac{20}{3} - \tfrac{2}{3}(n - 1) = \boxed{\frac{22 - 2n}{3}}.$$ □

Remark 1.5. In general, observe that

- The first two terms of an AP completely determine how it continues,
- More generally, if we know two terms and their corresponding positions in an AP, then the whole sequence can be recovered,
- Every AP has an nth term corresponding to a "straight line" function of n, i.e., an expression of the form $b + cn$. (These are not called *linear*, but *affine*—linear functions are the ones we covered in chapter 1).

Exercise 1.6. 1. Given that the following are all AP's, determine the term indicated in brackets, as well as their nth term.

a) $3, 7, 11, \ldots$ (19th)

b) $\_, \_, 5, -1, \ldots$ (20th)

c) $\_, \_, \frac{1}{2}, \_, \_, 3, \ldots$ (40th)

d) $\_, 2, \_, \_, \_, \_, 3, \ldots$ (1st)

e) $3, \_, \_, \pi, \_, \ldots$ (5th)

f) $\sqrt{2}, \sqrt{3}, \ldots$ (10th)

2. Consider following three APs.

a_n :	6,	_,	_,	21,	_,	_,	_,	...
b_n :	_,	_,	4,	_,	_,	25,	_,	...
c_n :	_,	12,	_,	_,	_,	44,	_,	...

Are there values of n such that a_n and b_n have the same term in the nth position? What about a_n and c_n, or b_n and c_n?

3. The fifth term is a, the sum of the 8th and the 6th term is
4. the product of consecutive terms
5. The average of terms
6. x, x^2 and x^3 are AP.

1.2 Geometric Sequences

In contrast to "arithmetic", the word "geometric" is usually synonymous with "multiplicative", i.e., it is an adjective describing something which involves multiplication. Indeed, an *geometric progression* (GP) is what we call a sequence where we always multiply by the same number to get from one