1 Propositional Logic (i.e., 0th Order Logic)

Statements or *propositions* are sentences which can be decidedly assigned a truth-value, i.e., **true** or **false**. Examples of statements include "Today is a rainy day" or "$1 + 1 = 5$", whereas examples of non-statements are "What time is it?", "Edam cheese", "73" or "This statement is false".[1]

For what we study here, it is not important how we are able to determine the truth-value of a statement; we only care that it can be done. In practice, we might require very different techniques in order to do so, depending on the nature of the statement (e.g., we would use different techniques for checking whether or not "Today is a rainy day" is true, than we would for checking "$1 + 1 = 5$"). Our goal is to study what happens when we combine statements together, developing a calculus which allows us to discover things about the truth-values of a compound proposition, assuming we know the truth-values of its individual component propositions.

For instance, suppose P and Q are both propositions, and that we know whether they are true or false. This is the only information about them which we will allow ourselves to use; we will not assume anything else about them. Given this, can we say anything about the compound statement "If P, then Q"? What about "P and not Q"?

Towards this goal, we now introduce the following *connectives* which allow us to construct compound propositions from other ones.

Definitions 1.1 (Logical connectives)**.** Let P and Q denote propositions.

(i) $\neg P$ denotes the statement "not P" called the *negation of* P, which is defined to be true precisely when P is false, and vice-versa.

(ii) $P \wedge Q$ denotes the statement "P and Q", called the *conjunction of* P *and* Q, which is defined to be true precisely when P and Q are both true, and false otherwise.

(iii) $P \vee Q$ denotes the statement "P or Q", called the *disjunction of* P *and* Q, which is defined to be false precisely when P and Q are both false, and true otherwise.

[1] This last one is interesting. Why is it not a statement? Well, statements must be either true or false; so let us suppose that it is true. We immediately see the contradiction that arises: it claims itself to be false, so it cannot be true. Suppose therefore that it is false. In this case, it makes a claim which is true — contradicting that it should be false. Thus we cannot say that the statement is true, nor that it is false. The problem with this statement is that it talks about itself; we need to be careful with how we deal with self-reference.

(iv) $P \to Q$ denotes the statement "if P, then Q" or "P implies Q", called the (*material*) *implication of P and Q*, which is defined to be false precisely when P true and Q is false, and true otherwise.

(v) $P \leftrightarrow Q$ denotes the statement "P if and only if Q" or "P iff Q" (for short), called the *bi-implication* or *biconditional of P and Q*, which is defined to be true precisely when P and Q are the same (i.e., both true or both false), and false otherwise.

Examples 1.2. Let P = "$1+1=2$", Q = "Pigs can fly" and R = "$7<3$". These are true, false and false respectively. We have:

(i) $\neg P$ is "$1+1 \neq 2$", which is false, since P is true. $\neg Q$ is "Pigs cannot fly", which is true, and $\neg R$ is "$7 \geqslant 3$", which is also true.

(ii) $P \wedge Q$ is "$1+1=2$ and pigs can fly", which is false, since Q is false.

(iii) $Q \vee R$ is "Pigs can fly or $7<3$". This is false, since both are false. On the other hand, $Q \vee P$ is "Pigs can fly or $1+1=2$". Since P is true, this is true.

(iv) $Q \to R$ is "If pigs can fly, then $7<3$". This is true since Q, i.e., "pigs can fly" is false. On the other hand, $P \to Q$ is "If $1+1=2$, then pigs can fly" is false, whereas $Q \to Q$ which is "If $1+1=2$ then $1+1=2$" is true.

(v) $Q \leftrightarrow R$ is "Pigs can fly if and only if $7<3$" is true, because Q and R are both false. $P \leftrightarrow \neg Q$ is "$1+1=2$ if and only if pigs cannot fly" is true.

1.1 Proof Theory

Our goal is to prove things about propositions. For instance, it turns out that if P is true, then $P \vee Q$ is always true. To prove something like this, we will introduce a list of *inference rules* which completely capture the behaviour of our connectives. When we write

$$\frac{P}{Q} \qquad \text{or} \qquad P \vdash Q,$$

we mean that "if we assume P, then we can prove Q by applying our rules". We call P a *hypothesis*, and Q a *conclusion*. Sometimes we need to assume multiple hypotheses to obtain a conclusion, in this case, we write

$$\frac{P_1\,,\, P_2\,,\, \ldots\,,\, P_n}{Q} \qquad \text{or} \qquad P_1\,,\, P_2\,,\, \ldots\,,\, P_n \vdash Q$$

or sometimes

$$\frac{\mathbf{\Gamma}}{Q} \qquad \text{or} \qquad \mathbf{\Gamma} \vdash Q,$$

where we use $\mathbf{\Gamma}$ to denote collection of hypotheses (say $\mathbf{\Gamma} = P_1, P_2, \ldots, P_n$). Notice that the symbols $\vdash$ and , do not live in the same "realm" as the logical connectives, they are what we call *meta-symbols*, and cannot appear in propositions; they are telling us *about* propositions. (If some of this is not clear, take a look at this very good Wikipedia article for a more detailed explanation. In particular, the sections "Explanation", "Terminology" and "Basic Concepts" should be very helpful.)

We will use our new notation to write our inference rules, which are given below. Each rule has a name associated with it, which we write to its right.

Truth and Falsehood

$$\frac{}{\textbf{true}}\ \textbf{true}\text{-int} \qquad \frac{\textbf{false}}{A}\ \textbf{false}\text{-elim}$$

Conjunction

$$\frac{A\,,\,B}{A \wedge B}\ \wedge\text{-int} \qquad \frac{A \wedge B}{A}\ \wedge\text{-elim}_1 \qquad \frac{A \wedge B}{B}\ \wedge\text{-elim}_2$$

Disjunction

$$\frac{A}{A \vee B}\ \vee\text{-int}_1 \qquad \frac{B}{A \vee B}\ \vee\text{-int}_2 \qquad \frac{A \rightarrow C\,,\,B \rightarrow C\,,\,A \vee B}{C}\ \vee\text{-elim}$$

Implication

$$\frac{A \rightarrow B\,,\,A}{B}\ \rightarrow\text{-elim} \qquad \frac{\mathbf{\Gamma}\,,\,A \vdash B}{\mathbf{\Gamma} \vdash A \rightarrow B}\ \rightarrow\text{-int}$$

Biconditional

$$\frac{A \rightarrow B\,,\,B \rightarrow A}{A \leftrightarrow B}\ \leftrightarrow\text{-int} \qquad \frac{A \leftrightarrow B}{A \rightarrow B}\ \leftrightarrow\text{-elim}_1 \qquad \frac{A \leftrightarrow B}{B \rightarrow A}\ \leftrightarrow\text{-elim}_2$$

Negation

$$\frac{\neg\neg A}{A}\ \neg\text{-elim} \qquad \frac{A \rightarrow B\,,\,A \rightarrow \neg B}{\neg A}\ \neg\text{-int}$$

We can think of these rules as a "step" in a proof. We can combine them together to make more interesting conclusions. For instance, we can show that $Q \wedge P$ can be obtained from $P \wedge Q$.

1	$P \wedge Q$	
2	P	($\wedge$-elim$_1$, 1)
3	Q	($\wedge$-elim$_2$, 1)
4	$Q \wedge P$	($\wedge$-int, 3, 2)

This is called a Fitch proof. Notice how we present the argument: each line is numbered, and we can apply the inference rules by referring to previous lines. The third column justifies how each line is obtained. By constructing a proof like this, we have show that we can prove $Q \wedge P$ by assuming $P \wedge Q$, i.e., that $P \wedge Q \vdash Q \wedge P$.

Let us give another example.

Example 1.3. We prove that $P \wedge Q \vdash P \wedge (Q \vee P)$.

1	$P \wedge Q$	(hypothesis)
2	P	($\wedge$-elim$_1$, 1)
3	Q	($\wedge$-elim$_2$, 1)
4	$Q \vee P$	($\vee$-int$_1$, 3)
5	$P \wedge (Q \vee P)$	($\wedge$-int 2, 4)

High-level idea: Just as we did in the first example, we can split the conjunction $P \wedge Q$ using the two $\wedge$-elim rules, and then add whatever we want to the solitary Q using $\vee$-int.

Then we can put things back together using the $\wedge$-int rule.

1.2 Subproofs

The scariest looking rule in our list is probably

$$\frac{\mathbf{\Gamma}, A \vdash B}{\mathbf{\Gamma} \vdash A \to B} \to\text{-int}.$$

What is this saying? (A rule like this is called a deduction theorem). It basically links our ability to prove things with them being true. Remember that

$$P \to Q$$

by definition means that if P is true, then Q is also true. In contrast,

$$P \vdash Q$$

means that if we start from P, we can apply our rules and end up with Q. Does this mean that Q is true? Well we hope so, otherwise the way our rules work isn't very helpful. In fact, the rule above actually guarantees that if $P \vdash Q$, then $P \to Q$. Indeed, what the rule says is that, if we assume a list of hypotheses $\mathbf{\Gamma}$, and adding A to that list allows us to prove B, then from $\mathbf{\Gamma}$ alone, we can deduce that $A \to B$. In a Fitch proof, this looks like this:

	$\vdots$	
10	$\quad$ | A	(subhypothesis)
	$\quad$ | $\vdots$	
14	$\quad$ | B	(some rule)
15	$A \to B$	($\to$-int, 10–14)
	$\vdots$	

We call the indented part a *subproof*. Lines deduced within a subproof can only be used (i.e., combined using inference rules) within that subproof, but lines outside a subproof can be used within it.

We now give some more examples. Important results which can be quoted in the exam are distinguished from other examples as theorems.

Example 1.4. We prove that $\vdash A \leftrightarrow A$.

1	$\quad \vert\; A$	(subhypothesis)
2	$A \to A$	($\to$-int, 1–1)
3	$A \leftrightarrow A$	($\leftrightarrow$-int, 2, 2)

High-level idea: We want to use $\leftrightarrow$-int, so we're going to need an implication $A \to A$. But this is easily obtained by taking A as a hypothesis and applying $\to$-int.

A very useful theorem is the following one.

Theorem 1.5 (Principle of explosion)**.** *For any two propositions P and Q, we have*

$$P \wedge \neg P \vdash Q.$$

This result is also sometimes referred to as the bivalence property or *ex falso quodlibet*. Here is the proof.

1	$P \wedge \neg P$	(hypothesis)
2	P	($\wedge$-elim$_1$, 1)
3	$\neg P$	($\wedge$-elim$_2$, 1)
4	$\quad \vert\; \neg Q$	(subhypothesis)
5	$\quad \vert\; P$	(line 2)
6	$\neg Q \to P$	($\to$-int, 4–5)
7	$\quad \vert\; \neg Q$	(subhypothesis)
8	$\quad \vert\; \neg P$	(line 3)
9	$\neg Q \to \neg P$	($\to$-int, 7–8)
10	$\neg\neg Q$	($\neg$-int, 6, 9)
11	Q	($\neg$-elim, 10)

High-level idea: The only rule which seems promising here is $\neg$-int, since we can use it to negate any hypothesis A by concluding both P and $\neg P$ from A (just by copying both of them in the respective subproofs).

If we let this hypothesis by $\neg Q$, the rule gives us $\neg\neg Q$, but this is equivalent to Q by $\neg$-elim.

Example 1.6 (Modus tollendo ponens)*.* We prove the *disjunctive syllogism*

$$p \vee q, \neg p \vdash q,$$

also known as modus tollendo ponens.

(In alternative formulations of the inference rules, this is sometimes given as the rule for $\vee$-elim. There isn't a unique set of deduction rules which one can state for propositional logic; we just pick a set which is as small as possible and try to prove everything else from them.)

1	$p \vee q$	(hypothesis)
2	$\neg p$	(hypothesis)
3	│ q ∟	(subhypothesis)
4	│ q	(line 3)
5	$q \rightarrow q$	($\rightarrow$-int, 3–4)
6	│ p ∟	(subhypothesis)
7	│ $p \wedge \neg p$	($\wedge$-int 6, 2)
8	│ q	(principle of explosion, 7)
9	$p \rightarrow q$	($\rightarrow$-int, 6–8)
10	q	($\vee$-elim, 9, 5, 1)

High-level idea: We want to use $\vee$-elim to end up with a solitary q. Since we have $p \vee q$, it might be simplest to use this in $\vee$-elim, then we just need to prove $p \rightarrow q$ and $q \rightarrow q$.

Obtaining the latter is straightforward (take q as a subhypothesis, copy it on the next line and apply $\rightarrow$-int). For the former, we know that $\neg p$ is true since it's one of our hypotheses, so assuming p should allow us to deduce anything using the principle of explosion (including q).

We can actually do the proof without the principle of explosion with a little bit more work.

1	$p \vee q$	(hypothesis)
2	$\neg p$	(hypothesis)
3	│ p ∟	(subhypothesis)
4	│ │ $\neg q$ ∟	(subsubhypothesis)
5	│ │ $\neg p$	(line 2)
6	│ $\neg q \rightarrow \neg p$	($\rightarrow$-int, 4–5)
7	│ │ $\neg q$ ∟	(subsubhypothesis)
8	│ │ p	(line 3)
9	│ $\neg q \rightarrow p$	($\rightarrow$-int, 7–8)
10	│ $\neg\neg q$	($\neg$-int, 9, 6)
11	│ q	($\neg$-elim, 10)
12	$p \rightarrow q$	($\rightarrow$-int, 3–11)
13	│ q ∟	(subhypothesis)
14	│ q	(line 13)
15	$q \rightarrow q$	($\rightarrow$-int, 13–14)
16	q	($\vee$-elim, 12, 15, 1)

We can show that $\neg q$ implies both p and $\neg p$ (just by copying them from their respective line numbers).

This will lead to $\neg\neg q$ by $\neg$-int.

Removing the double negation with $\neg$-elim will complete the proof that $p \rightarrow q$.

(The idea here is quite similar to the proof of principle of explosion itself.)

Exercise 1.7. 1. Explain the difference between $p \to q$ and $p \vdash q$.

2. Construct a Fitch proof for the following.

   (a) $P \vee \textbf{false} \vdash P$

   (b) $P \vdash Q \to P$

   (c) $P \vdash \neg\neg P$

   (d) $P \vee (Q \wedge R) \vdash (P \vee Q) \wedge (P \vee R)$

   (e) $(P \wedge \neg Q) \vee (\neg P \wedge Q) \vdash \neg(P \leftrightarrow Q)$

   (f) $P \to Q \dashv\vdash \neg Q \to \neg P$

   (g) $\neg P \wedge \neg Q \vdash \neg(P \vee Q)$

3. *Epistemic logic* is an extension of propositional logic by the pair of unary connectives $\Box$ (necessity) and $\Diamond$ (possibility). We use this logic to talk about what a person X knows. Intuitively, $\Box p$ means that X believes that p is true, whereas $\Diamond p$ means that X believes that p is possible.

   Their behaviour is determined by these rules.

$$\frac{\Box(A \to B)}{\Box A \to \Box B}\,\mathbf{K} \qquad \frac{\Box A}{\Box\Box A}\,\mathbf{4} \qquad \frac{\neg\Box A}{\Box\neg\Box A}\,\mathbf{5}$$

$$\frac{A}{\Box\Diamond A}\,\mathbf{B} \qquad \frac{\Box A}{\Diamond A}\,\mathbf{D} \qquad \frac{\Box A}{\neg\Diamond\neg A}\,\text{dual}_1 \qquad \frac{\Diamond A}{\neg\Box\neg A}\,\text{dual}_2$$

   Notice that just because X knows something, doesn't mean that it is true. (In other words, $\Box A \vdash A$ is not a rule.)

   (a) Translate each of the rules **K**, **4**, **5**, **B**, **D**, dual_1 and dual_2 into English so that you understand them better. For instance, dual_1 becomes

      If X thinks that A is true, then he doesn't believe that A is not possible.

   (b) Construct a Fitch proof for the following.

      i. $P \vdash \Diamond\Diamond P$

      ii. $\Box P, \Box(P \to Q) \vdash \Box Q$ (Modus Ponens)

      iii. $\Box(P \to Q) \vdash \Diamond P \to \Diamond Q$

      Now adding the rule $\frac{A}{\Box A}\,\mathbf{G}$,

      iv. $\Box A \wedge \Box B \vdash \Box(A \wedge B)$

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Another very useful theorem is the following.

Theorem 1.8 (Law of the Excluded Middle)**.** *For any proposition P, we have*

$$\vdash P \vee \neg P.$$

The proof is not very straightforward.

1	$\neg(P \vee \neg P)$	(subhypothesis)
2	$\quad P$	(subsubhypothesis)
3	$\quad P \vee \neg P$	($\vee$-int$_1$, 2)
4	$P \to P \vee \neg P$	($\to$-int, 2–3)
5	$\quad P$	(subsubhypothesis)
6	$\quad \neg(P \vee \neg P)$	(line 1)
7	$P \to \neg(P \vee \neg P)$	($\to$-int, 5–6)
8	$\neg P$	($\neg$-int, 4, 7)
9	$P \vee \neg P$	($\vee$-int$_2$, 8)
10	$\neg(P \vee \neg P) \to P \vee \neg P$	($\to$-int, 1–9)
11	$\neg(P \vee \neg P)$	(subhypothesis)
12	$\neg(P \vee \neg P)$	(line 11)
13	$\neg(P \vee \neg P) \to \neg(P \vee \neg P)$	($\to$-int, 11–12)
14	$\neg\neg(P \vee \neg P)$	($\neg$-int, 10, 13)
15	$P \vee \neg P$	($\neg$-elim, 14)

High-level idea: Since we have no hypotheses, we must start with a subhypothesis, but the conclusion cannot contain any $\to$, so we try to look at rules which get rid of $\to$'s: $\neg$-int looks promising.

If we can conclude $\neg\neg(P \vee \neg P)$, we can then use $\neg$-elim to finish.

Clearly $\neg(P \vee \neg P)$ implies itself, so if we can also show it implies $P \vee \neg P$, we are done. This is what we do in 1–9.

A very useful result is the following.

Theorem 1.9 (Law of Contrapositive)**.** *For any propositions P and Q, we have*

$$P \to Q \dashv\vdash \neg Q \to \neg P.$$

Two other very useful (dual) results are de Morgan's laws.

Theorem 1.10 (de Morgan's Laws)**.** *For any two propositions P and Q, we have*

$$\neg(P \wedge Q) \dashv\vdash \neg P \vee \neg Q \qquad \textit{and} \qquad \neg(P \vee Q) \dashv\vdash \neg P \vee \neg Q.$$

These allow us to switch between $\wedge$ and $\vee$. There are four proofs required here, $\vdash$ and $\dashv$ for both of them.