# Legendre's Equation and Legendre Polynomials

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## 1 Introduction

In these notes, we solve Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{1}$$

using the method of power series, and then we subsequently define Legendre polynomials and explore some of their properties.

### 1.1 Overview of the Power Series Method

Most differential equations have solutions which cannot be described using elementary functions (i.e. polynomials, trigonometric functions, logarithms, and so on; functions we commonly work with). In fact, even the solutions of the simple differential equation $y'' + xy = 0$, known as Airy's equation, cannot be written in terms of elementary functions.

We can however write its solutions using power series:

$$y = A\sum_{n=0}^{\infty} \frac{(-1)^n \prod_{k=0}^{n-1}(3k+1)}{(3n)!}x^{3n} + B\sum_{n=0}^{\infty} \frac{(-1)^n \prod_{k=0}^{n-1}(3k+2)}{(3n+1)!}x^{3n+1}, \tag{2}$$

for any $A, B \in \mathbb{R}$.

Even though the solutions look complicated in this case, all we care about is the fact that they can be represented as power series; i.e. functions of the form $\sum_{n=0}^{\infty} a_n x^n$ where $a_0, a_1, \ldots$ are the coefficients.

**Example 1** (Simple Harmonic Motion)**.** Consider the equation

$$y'' + y = 0. \tag{3}$$

Suppose the solution can be written as a power series, say, $y = \sum_{n=0}^{\infty} a_n x^n$. Then differentiating term by term, we get that the first and second derivatives of $y$ are given by the power series

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2}. \tag{4}$$

Substituting these in equation (3), we have

$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0,$$

then by replacing $n$ with $n+2$ everywhere in the first series, we get that both series have the same power of $x$ in their general term:

$$\implies \sum_{n=-2}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Now expanding out the first few terms of the first series, we can combine the two summations since the bottom indices match:

$$\implies a_0(0)(-1)x^{-2} + a_1(1)(0)x^{-1} + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\implies \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\implies \sum_{n=0}^{\infty} (a_{n+2}(n+2)(n+1) + a_n)x^n = 0.$$

Since this series must be zero (for any value of $x$), each of the coefficients must be zero; i.e. we must have

$$\begin{aligned} a_{n+2}(n+2)(n+1) + a_n &= 0 \\ \implies a_{n+2} &= -\frac{a_n}{(n+2)(n+1)}. \end{aligned}$$

Thus the first few coefficients are

$$a_0, a_1, a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{3 \cdot 2}, a_4 = \frac{a_0}{4 \cdot 3 \cdot 2}, a_5 = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}, a_6 = -\frac{a_0}{6!}, \dots,$$

so the solutions are given by

$$\begin{aligned} y(x) &= a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 - \frac{a_0}{6!}x^6 - \cdots \\ &= a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= a_0 \cos x + a_1 \sin x. \end{aligned}$$

**Example 2** (Hermite's Equation)**.** Consider the equation

$$y'' - x^2 y = 0 \tag{5}$$

Suppose the solution can be written as a power series: $y = \sum_{n=0}^{\infty} a_n x^n$. Then the first and second derivatives are given by $y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2}$. Substituting these in equation (5), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0 \\ \implies &\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \\ \implies &\sum_{n=-4}^{\infty} a_{n+4}(n+4)(n+3)x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \qquad (n \leftarrow n+4) \\ \implies &\underbrace{0 + 0 + 2a_2 + 6a_3 x}_{\text{terms for } n = -4, \dots, -1} + \sum_{n=0}^{\infty} a_{n+4}(n+4)(n+3)x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \\ \implies &2a_2 + 6a_3 x + \sum_{n=0}^{\infty} (a_{n+4}(n+4)(n+3) - a_n)x^{n+2} = 0 \end{aligned}$$

Just as in the last example, we have that all coefficients must be zero. In particular, $2a_2 = 0$, which means that $a_2 = 0$, similarly $6a_3 = 0$, so $a_3 = 0$, and

$$a_{n+4}(n+4)(n+3) - a_n = 0 \implies a_{n+4} = \frac{a_n}{(n+4)(n+3)}.$$

Thus we have the coefficients

$$a_0, a_1, a_2 = a_3 = 0, a_4 = \frac{a_0}{4 \cdot 3}, a_5 = \frac{a_1}{5 \cdot 4}, a_6 = a_7 = 0, a_8 = \frac{a_0}{8 \cdot 7 \cdot 4 \cdot 3}, \dots$$

so the solutions are given by

$$\begin{aligned} y(x) &= a_0 + a_1 x + \frac{a_0}{4 \cdot 3}x^4 + \frac{a_1}{5 \cdot 4}x^5 + \frac{a_0}{8 \cdot 7 \cdot 4 \cdot 3}x^8 + \frac{a_1}{9 \cdot 8 \cdot 5 \cdot 4}x^9 + \cdots \\ &= a_0 \left(1 + \frac{x^4}{4 \cdot 3} + \frac{x^8}{8 \cdot 7 \cdot 4 \cdot 3} + \cdots\right) + a_1 \left(x + \frac{x^5}{5 \cdot 4} + \frac{x^9}{9 \cdot 8 \cdot 5 \cdot 4} + \cdots\right) \\ &= a_0 \sum_{n=0}^{\infty} \frac{x^{4n}}{\prod_{k=1}^{n} 4k(4k-1)} + a_1 \sum_{n=0}^{\infty} \frac{x^{4n+1}}{\prod_{k=1}^{n}(4k+1)4k}, \end{aligned}$$

where this time, the power series do not correspond to functions we can immediately recognise. In fact, these two power series are used to define what are known as Hermite functions.[1]

**Remark 3.** The general power series method to solve

$$a(x)y'' + b(x)y' + c(x)y = 0$$

is to let $y(x) = \sum_{n=0}^{\infty} a_n x^n$, then $y'$ and $y''$ are as in (4). Substitute these into the equation, and simplify to get the left-hand side as a single power series. Since the series must be zero independently of $x$, then each coefficient must be zero, which gives a *recurrence relation* for the coefficients $a_n$. This gives two linearly independent solutions for $y(x)$.

The power series method always works so long as for all $x \in \mathbb{R}$, $a(x) \neq 0$. If $a(x) = 0$ for some values of $x$, then the so-called Frobenius method can be used, where we instead take $y = x^r \sum_{n=0}^{\infty} a_n x^n$ with $a_0 \neq 0$ and proceed similarly.

[1] It is not uncommon in mathematics to *define* functions by differential equations. For example, $\cos x$ can be defined as the unique solution of $y'' + y = 0$ (equation (3)) with $y(0) = 1$ and $y'(0) = 0$ (since these conditions force $a_0$ to be 1 and $a_1$ to be 0).

**Exercise 4.** Verify, using the method of power series, that the solutions of Airy's equation $y'' + xy = 0$ are given by

$$y = a_0\left(1 - \frac{1}{3!}x^3 + \frac{4}{6!}x^6 - \frac{7\cdot 4}{9!}x^9 + \cdots\right) + a_1\left(x - \frac{2}{4!}x^4 + \frac{5\cdot 2}{7!}x^7 - \cdots\right)$$

as is stated in equation (2).

The two linearly independent power series solutions which arise from this differential equation are called *Airy's functions*, denoted $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$.

## 2 Legendre's Equation

Now we use the power series method to solve Legendre's equation

$$(1-x^2)y'' - 2xy' + \xi(\xi+1)y = 0, \tag{6}$$

where $\xi \in \mathbb{R}$ is a constant.

As usual, suppose $y = \sum_{n=0}^{\infty} a_n x^n$. Then the first and second derivatives are given by $y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2}$, and substituting these in equation (6), we have

$$(1-x^2)\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 2x\sum_{n=0}^{\infty} a_n n x^{n-1} + \xi(\xi+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\implies \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n n(n-1)x^n - \sum_{n=0}^{\infty} 2a_n n x^n + \sum_{n=0}^{\infty} a_n \xi(\xi+1)x^n = 0$$

$$\implies \sum_{n=-2}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n n(n-1)x^n - \sum_{n=0}^{\infty} 2a_n n x^n + \sum_{n=0}^{\infty} a_n \xi(\xi+1)x^n = 0$$

$$\implies 0 + 0 + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n n(n-1)x^n - \sum_{n=0}^{\infty} 2a_n n x^n + \sum_{n=0}^{\infty} a_n \xi(\xi+1)x^n = 0$$

$$\implies \sum_{n=0}^{\infty}(a_{n+2}(n+2)(n+1) - a_n n(n-1) - 2a_n n + a_n \xi(\xi+1))x^n = 0$$
$$\implies \sum_{n=0}^{\infty}(a_{n+2}(n+2)(n+1) + (\xi+n+1)(\xi-n)a_n)x^n = 0,$$

and since the coefficients must be zero, we have

$$a_{n+2} = -\frac{(n+\xi+1)(\xi-n)}{(n+2)(n+1)}\, a_n,$$

so the power series coefficients of $y$ are

$$a_0, a_1, a_2 = -\frac{(\xi+1)\xi}{2\cdot 1}\, a_0, a_3 = -\frac{(\xi+2)(\xi-1)}{3\cdot 2}\, a_1,$$
$$a_4 = \frac{(\xi+3)(\xi-2)(\xi+1)\xi}{4\cdot 3\cdot 2\cdot 1}\, a_0, a_5 = \frac{(\xi+4)(\xi-3)(\xi+2)(\xi-1)}{5\cdot 4\cdot 3\cdot 2}\, a_1, \ldots,$$

and the solutions are given by

$$\begin{aligned} y &= a_0 + a_1 x - \frac{(\xi+1)\xi}{2!} a_0 x^2 - \frac{(\xi+2)(\xi-1)}{3!} a_1 x^3 + \cdots \\ &= a_0\left(1 - \frac{(\xi+1)\xi}{2!}x^2 + \cdots\right) + a_1\left(x - \frac{(\xi+2)(\xi-1)}{3!}x^3 + \cdots\right) \\ &= a_0\, y_0(\xi, x) + a_1\, y_1(\xi, x), \end{aligned}$$

where

$$\begin{aligned} y_0(\xi, x) &= 1 - \frac{(\xi+1)\xi}{2!}x^2 + \frac{(\xi+3)(\xi-2)(\xi+1)\xi}{4!}x^4 - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{k=0}^{2n-1}(\xi - k(-1)^k)}{(2n)!}\, x^{2n}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} y_1(\xi, x) &= x - \frac{(\xi+2)(\xi-1)}{3!}x^3 + \frac{(\xi+4)(\xi-3)(\xi+2)(\xi-1)}{5!}x^5 - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{k=0}^{2n-1}(\xi - (-1)^k(k+1))}{(2n+1)!}\, x^{2n+1}. \end{aligned} \tag{8}$$

**Exercise 5.** Use the power series method to show that the solutions of the special case $(1-x^2)y'' - 2xy' + 2y = 0$ of Legendre's equation are given by $y(x) = a_0 x + a_1(1 - x\tanh^{-1} x)$.

# 3 Legendre Polynomials

Observe that the two linearly independent functions $y_0(n,x)$ and $y_1(n,x)$ from equations (7) and (8) terminate, that is, consist of finitely many terms, if $n$ is an even/odd non-negative integer respectively.

Indeed, if $n \geqslant 0$ is even, then after finitely many terms, the bracket $(\xi - n) = (n-n) = 0$ appears in all numerators in the series expansion of $y_0(n,x)$, making them all zero. This does not happen for $y_1(n,x)$ if $n$ is even, since in $y_1$ we get the bracket $(\xi + n) = (n+n) = 2n$. For example,

$$\begin{aligned} y_0(4,x) &= 1 - 10x^2 + \frac{35}{3}x^4 + 0x^6 + 0x^8 + \cdots = 1 - 10x^2 + \frac{35}{3}x^4 \\ y_1(4,x) &= x - 3x^3 + \frac{6}{5}x^5 + \frac{2}{7}x^7 + \frac{1}{7}x^9 + \frac{1}{11}x^{11} + \frac{28}{429}x^{13} + \cdots . \end{aligned}$$

If, on the other hand, $n$ is odd, then the same happens for $y_1(n,x)$ since the bracket $(\xi - n)$ appears in every numerator after finitely many terms, but it does not happen for $y_0(n,x)$. For example,

$$\begin{aligned} y_0(5,x) &= 1 - 15x^2 + 30x^4 - 10x^6 - \frac{15}{7}x^8 - x^{10} - \frac{20}{33}x^{12} + \cdots \\ y_1(5,x) &= x - \frac{14}{3}x^3 + \frac{21}{5}x^5 + 0x^7 + 0x^9 + \cdots = x - \frac{14}{3}x^3 + \frac{21}{5}x^5. \end{aligned}$$

Observe that in either case, the polynomial which terminates has degree $n$.

**Definition 6** (Legendre Polynomial)**.** Let $n$ be a non-negative integer. The *Legendre polynomial of order* $n$ is the polynomial $P_n(x)$ of degree $n$ satisfying the Legendre equation $y'' - 2xy' + n(n+1)y = 0$ and $P_n(1) = 1$.

In view of the observations above, the Legendre polynomial is given by

$$P_n(x) = \begin{cases} a_0\, y_0(n,x) & \text{if } n \text{ is even} \\ a_1\, y_1(n,x) & \text{if } n \text{ is odd,} \end{cases}$$

where the constant $a_0$ (or $a_1$) is chosen appropriately so that $P_n(1) = 1$. (Indeed, if $y_i(n,1) \neq 1$, then we simply divide by $y_i(n,1)$). Thus

$$P_n(x) = \begin{cases} \dfrac{y_0(n,x)}{y_0(n,1)} & \text{if } n \text{ is even} \\[2ex] \dfrac{y_1(n,x)}{y_1(n,1)} & \text{if } n \text{ is odd.} \end{cases}$$

**Example 7.** We determine the Legendre polynomials $P_5(x)$ and $P_6(x)$.

Indeed, by equation (8), we have $y_1(5,x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5$, and therefore $y_1(5,1) = 1 - \frac{14}{3} + \frac{21}{5} = \frac{8}{15}$. Thus $P_5(x) = \frac{1}{8/15}\, y_1(5,x) = \frac{15}{8}(x - \frac{14}{3}x^3 + \frac{21}{5}x^5) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$.

By equation (7), we have $y_0(6,x) = 1 - 21x^2 + 63x^4 - \frac{231}{5}x^6$. Consequently, $y_0(6,1) = 1 - 21 + 63 - \frac{231}{5} = -\frac{16}{5}$. Hence $P_6(x) = \frac{1}{-16/5}\, y_0(6,x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$.

**Exercise 8.** Determine the Legendre polynomials $P_8(x)$, $P_9(x)$ and $P_{10}(x)$.

# 4 Properties of Legendre Polynomials

Here we illustrate some of the nice properties which Legendre polynomials exhibit.

## 4.1 Rodrigues' Formula

**Theorem 9** (O. Rodrigues, 1816 [1])**.** Let $n$ be a non-negative integer. Then

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n.$$

*Proof.* Let $\phi(x)$ denote the right-hand side. It suffices to show that $\phi(x)$ is a polynomial which satisfies the Legendre equation (6), and that $\phi(1) = 1$.

That $\phi(x)$ is a polynomial is obvious, being a constant multiple of repeated derivatives of a polynomial, and can easily be shown by induction. Now for

$\phi(1) = 1$, one only need observe that $(x^2-1)^n = (x+1)^n(x-1)^n$, and then by Leibniz,[2]

$$\begin{aligned}\phi(x) &= \frac{1}{n!\,2^n}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}(x+1)^n\frac{d^{n-k}}{dx^{n-k}}(x-1)^n\right)\\ &= \frac{1}{n!\,2^n}\left((x+1)^n n! + \sum_{k=1}^{n}\binom{n}{k}\frac{d^k}{dx^k}(x+1)^n\frac{d^{n-k}}{dx^{n-k}}(x-1)^n\right),\end{aligned}$$ [3]

so that when $x = 1$, $\phi(1) = \frac{1}{n!\,2^n}(2^n\,n! + \sum_{k=1}^{n} 0) = 1.$[4]

Finally to show that $\phi(x)$ satisfies the Legendre equation, let $\psi(x) = (1-x^2)^n$. Then $\psi'(x) = -2nx(1-x^2)^{n-1}$, and multiplying both sides by $(1-x^2)$ we get $(1-x^2)\psi' = -2nx(1-x^2)^n = -2nx\psi$, so that

$$(1-x^2)\psi' + 2nx\psi = 0.$$

Differentiating $n+1$ times, by Leibniz, we have

$$\begin{aligned}(1-x^2)\psi^{(n+2)} + (n+1)(-2x)\psi^{(n+1)} + \frac{(n+1)n}{2}&(-2)\psi^{(n)}\\ &+ 2nx\psi^{(n+1)} + 2n(n+1)\psi^{(n)} = 0,\end{aligned}$$

which simplifies to

$$\begin{aligned}&(1-x^2)\psi^{(n+2)} - 2x\psi^{(n+1)} + n(n+1)\psi^{(n)} = 0\\ \implies &(1-x^2)(\psi^{(n)})'' - 2x(\psi^{(n)})' + n(n+1)\psi^{(n)} = 0\end{aligned}$$

Now observe that $\psi^{(n)} = n!2^n(-1)^n\phi(x)$. Denote $n!2^n(-1)^n$ by $k$. Then we have

$$\begin{aligned}&(1-x^2)(k\phi)'' - 2x(k\phi)' + n(n+1)(k\phi) = 0\\ \implies &(1-x^2)k(\phi)'' - 2xk(\phi)' + n(n+1)(k\phi) = 0\\ \implies &(1-x^2)\phi'' - 2x\phi' + n(n+1)\phi = 0,\end{aligned}$$

so $\phi(x)$ satisfies Legendre's equation. □

[2] Leibniz rule for the $n$th derivative of a product: $(f\cdot g)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}f^{(k)}g^{(n-k)}$ (proof by induction).

[3] Here we use the intuitive fact that $\frac{d^n}{dx^n}(x+1)^n = n!$, easily proved by induction.

[4] Another intuitive fact provable by induction: if $x = 1$, then $\frac{d^{n-k}}{dx^{n-k}}(x-1)^n = 0$ for $1 \leqslant k \leqslant n$.

**Exercise 10.** Use Rodrigues' formula to obtain the fourth order Legendre polynomial $P_4(x)$.

**Exercise 11.** Prove the following results by induction to fill in the missing details of the proof.

(i) *Leibniz rule*: $(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$.

(ii) $\frac{d^n}{dx^n}(x+1)^n = n!$.

(iii) If $x = 1$, then $\frac{d^{n-k}}{dx^{n-k}}(x-1)^n = 0$ for $1 \leqslant k \leqslant n$.

## 4.2 Orthogonality

Sturm-Liouville theory is the study of second order differential equations of the form

$$(py')' + qy = -\lambda\omega y,$$

where $p, q, \omega$ are non-negative functions of $x$ in some interval, and $\lambda$ is a constant. The Legendre equation is of this form since $(1-x^2)' = -2x$, so it is equivalent to

$$((1-x^2)y')' + n(n+1)y = 0, \tag{9}$$

and for $x \in [-1, 1]$, $p(x) = 1 - x^2$ is non-negative.

If one considers the space of all real-valued functions on $[-1, 1]$ with the usual inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$, then we have the following.

**Theorem 12.** The Legendre polynomials are orthogonal, that is,

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$$

for $n \neq m$. When $n = m$, we have $\int_{-1}^{1} [P_n(x)]^2\,dx = 2/(2n+1)$.

*Proof.* Consider first $m \neq n$. Since these satisfy Legendre's equation, we have

$$((1-x^2)P_m')' + m(m+1)P_m = 0 \tag{10}$$

$$((1-x^2)P_n')' + n(n+1)P_n = 0 \tag{11}$$

using the Sturm-Liouville form. Doing $P_n(10) - P_m(11)$ and simplifying yields

$$[(1-x^2)(P_m'P_n - P_mP_n')]' + (m-n)(m+n+1)P_mP_n = 0.$$

Now integrating both sides, we get

$$\int_{-1}^{1} [(1-x^2)(P_m'P_n - P_mP_n')]'\,dx + (m-n)(m+n+1)\int_{-1}^{1} P_mP_n\,dx = 0,$$

and since the fist integral is clearly zero, we get $\int_{-1}^{1} P_mP_n\,dx = 0$ when $m \neq n$. Now for the result when $n = m$, we use Rodrigues' formula. Indeed,

$$\int_{-1}^{1} [P_n]^2\,dx = \frac{1}{(n!)^2\,2^{2n}} \underbrace{\int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^n}{dx^n}(x^2-1)^n\,dx}_{I_n}, \tag{12}$$

and proceeding by parts ($\int uv'\,dx = uv - \int vu'\,dx$), the integral $I_n$ becomes

$$\frac{d^n}{dx^n}(x^2-1)^n \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \bigg|_{-1}^{1} - \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n\,dx$$

and the first term vanishes by exercise 13(ii) with $k = 1$, so

$$I_n = -\int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n\,dx.$$

We can similarly integrate by parts again to obtain that

$$I_n = \int_{-1}^{1} \frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n \frac{d^{n+2}}{dx^{n+2}}(x^2-1)^n\,dx,$$

and similarly by induction, we can integrate by parts a total of $n$ times to get that

$$I_n = \int_{-1}^{1} (x^2-1)^n \frac{d^{2n}}{dx^{2n}}(x^2-1)^n\,dx = (2n)! \underbrace{\int_{-1}^{1} (x^2-1)^n\,dx}_{J_n} \tag{13}$$

by exercise 13(i). Now we substitute $t = (x+1)/2$ to transform the integral $J_n$ into

$$J_n = \int_0^1 t^n(1-t)^n\,dt.$$

Integrating this by parts yields the reduction formula $2(1+2n)J_n = nJ_{n-1}$. This can be solved to give that

$$J_n = \frac{(n!)^2}{(2n+1)!}$$

which when substituted back into equations (12) and (13) gives the desired result that $\int_{-1}^{1} [P_n(x)]^2 \, dx = 2/(2n+1)$. □

**Exercise 13.** Prove the following results by induction to fill in the missing details of the proof.

(i) $\frac{d^{2n}}{dx^{2n}}(x^2-1)^n = (2n)!$.

(ii) If $x = \pm 1$, then $\frac{d^{n-k}}{dx^{n-k}}(x^2-1)^n = 0$ for $1 \leqslant k \leqslant n$.

(iii) Suppose that $u^{(n-k)}(a) = u^{(n-k)}(b) = 0$ for $1 \leqslant k \leqslant n$. then

$$\int_a^b (u^{(n)}(x))^2 \, dx = \int_a^b u(x)\, u^{(2n)}(x) \, dx.$$

(iv) If $J_n = \int_0^1 t^n(1-t)^n \, dt$, then $2(1+2n)J_n = nJ_{n-1}$ and $J_0 = 1$. [Hint: integration by parts]

(v) If $J_0 = 1$ and $2(1+2n)J_n = nJ_{n-1}$, then $J_n = (n!)^2/(2n+1)!$.

**Exercise 14.** State the polynomials $P_4(x)$ and $P_5(x)$. Manually verify that $\int_{-1}^{1} P_4(x)P_5(x) \, dx = 0$ and that $\int_{-1}^{1}(P_4(x))^2 dx = \frac{2}{9}$.

## 4.3 Generalised Fourier Series (Fourier-Legendre)

**Exercise 15.** (i) Find constants $c_i \in \mathbb{R}$ so that the quadratic function $f(x) = a + bx + cx^2$ is equivalent to the combination of Legendre polynomials $c_0P_0(x) + c_1P_1(x) + c_2P_2(x)$.

In applications, it is often necessary to express a given real-valued function $f\colon [-1,1] \to \mathbb{R}$ in terms of Legendre polynomials, as done in exercise 15. By Sturm-Liouville theory, one can show that any square integrable[5] function

[5] Meaning that $\int_{-1}^{1} [f(x)]^2 \, dx$ exists.

$f$ on $[-1,1]$ with finitely many discontinuities can be expressed as a sum of Legendre polynomials:

$$f(x) \sim \sum_{n=0}^{\infty} c_n\, P_n(x), \tag{14}$$

where the relation $\sim$ becomes equality at points where $f$ is continuous, but where $f$ is discontinuous, we have

$$\sum_{n=0}^{\infty} c_n\, P_n(x) = \frac{1}{2}\left( \lim_{\xi \to x^-} f(\xi) + \lim_{\xi \to x^+} f(\xi) \right). \tag{15}$$

This series is called a *generalised Fourier series*, or in particular, the *Fourier-Legendre series* of $f$.

It is not hard to formally determine the coefficients $c_n$ in the general case thanks to the orthogonality property of the Legendre polynomials. Indeed, let us focus on the uncountably many points where $f(x)$ is continuous, so that the relation (14) is an equality. Multiplying both sides by $P_m(x)$ and integrating, we get

$$\int_{-1}^{1} f(x)P_m(x)\,dx = \sum_{n=0}^{\infty} c_n \int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2c_m}{2m+1},$$

by orthogonality (theorem 12). Thus we get that

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)P_n(x)\,dx.$$

Here we are of course assuming that $f(x)$ *can* be written as such a series in the first place. For a full proof, we direct the reader to section 4.7 of [2].

**Example 16** (Polynomials)**.** Let $f\colon [-1,1] \to \mathbb{R}$ be the polynomial of degree $d$ with coefficients $a_n$:

$$f(x) = \sum_{n=0}^{d} a_n x^n.$$

Clearly $f$ is square integrable and continuous everywhere, so we can express $f$ as $\sum_{n=0}^{\infty} c_n P_n(x)$ for all $x \in [-1,1]$. In this case, just as in exercise 15, we do not need to use the integral formula for $c_n$, one can obtain $c_i$ explicitly by solving linear equations arising from comparing coefficients, and then simply take $c_i = 0$ for $i \geqslant d+1$.

**Example 17.** Take $\alpha \in [-1, 1]$, and consider the function $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } -1 \leqslant x < \alpha \\ 1 & \text{if } \quad \alpha < x \leqslant 1. \end{cases}$$

Clearly $f$ is square integrable and has only one discontinuity (at $x = \alpha$), so we have that $f$ has Fourier-Legendre coefficients

$$\begin{aligned} c_n &= \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx \\ &= \frac{2n+1}{2} \left( \int_{-1}^{\alpha} 0 \cdot P_n(x)\, dx + \int_{\alpha}^{1} 1 \cdot P_n(x)\, dx \right) \\ &= \frac{2n+1}{2} \int_{\alpha}^{1} P_n(x)\, dx \\ &= \frac{1}{2} \int_{\alpha}^{1} (P'_{n+1} - P'_{n-1})\, dx \qquad \text{(assuming } n \geqslant 1\text{, using exercise 20(iv))} \\ &= \frac{1}{2}(P_{n+1}(1) - P_{n-1}(1) + P_{n+1}(\alpha) - P_{n-1}(\alpha)) = \frac{P_{n+1}(\alpha) - P_{n-1}(\alpha)}{2} \end{aligned}$$

since $P_n(1) = 1$ for all $n$. When $n = 0$, we have

$$c_0 = \frac{1}{2} \int_{-1}^{1} f(x) P_0(x)\, dx = \frac{1}{2} \int_{-1}^{1} f(x)\, dx = \frac{1}{2} \int_{-1}^{\alpha} 0\, dx + \frac{1}{2} \int_{\alpha}^{1} dx = \frac{1}{2}(1 - \alpha).$$

Thus we have that

$$f(x) = \frac{1}{2}(1-\alpha) P_0(x) + \frac{1}{2} \sum_{n=1}^{\infty} (P_{n+1}(\alpha) - P_{n-1}(\alpha))\, P_n(x),$$

for all $x \in [-1, 1]$ at which $f(x)$ is continuous. Now let us verify equation (15) for $f(x)$ at the discontinuity $x = \alpha$. Clearly we have $\lim_{x\to\alpha^-} f(x) = 0$ and $\lim_{x\to\alpha^+} f(x) = 1$, so we expect that the series above converges to $\frac{0+1}{2} = \frac{1}{2}$. Let $S_N$ denote the result of taking $N$ terms in the series obtained above. Indeed, when $x = \alpha$, we have

$$S_n = \frac{1}{2}(1-\alpha) P_0(\alpha) + \frac{1}{2} \sum_{n=1}^{N} (P_{n+1}(\alpha) - P_{n-1}(\alpha))\, P_n(\alpha)$$

$$\begin{aligned}
&= \frac{1}{2}(1-\alpha) + \frac{1}{2}\sum_{n=1}^{N} P_{n+1}(\alpha)P_n(\alpha) - P_n(\alpha)P_{n-1}(\alpha) \\
&= \frac{1}{2} - \frac{1}{2}P_{N+1}(\alpha)P_N(\alpha) \qquad \text{(method of differences)}
\end{aligned}$$

and as $N \to \infty$, we have that $P_N(x) \to 0$ (see the asymptotic form of $P_n(x)$ in section 4.6), so that $S_N \to \frac{1}{2}$, as expected.

**Exercise 18.** Let $f\colon [-1,1] \to \mathbb{R}$ be defined by

$$f(x) = \sqrt{\frac{1-x}{2}}.$$

(i) By multiplying both sides of the equation

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

by $f(x)$ and integrating both sides with respect to $x$, show that

$$\frac{1}{2t}\left(1 + t - \frac{(1-t)^2}{2\sqrt{t}}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right)\right) = \sum_{n=0}^{\infty} t^n \int_{-1}^{1} f(x)P_n(x)\,dx.$$

(ii) Expand the left-hand side of the equation given in (i) as a Maclaurin series in power of $t$ to obtain

$$\frac{4}{3} - 4\sum_{n=1}^{\infty} \frac{t^n}{(4n^2-1)(2n+3)} = \sum_{n=0}^{\infty} t^n \int_{-1}^{1} f(x)P_n(x)\,dx.$$

(iii) By comparing coefficients of $t^n$, deduce the values of $\int_{-1}^{1} f(x)P_n(x)\,dx$ for $n = 0$ and for $n \geqslant 1$.

(iv) Hence deduce that $f(x)$ can be written as the Fourier-Legendre series

$$f(x) = \frac{2}{3}P_0(x) - 2\sum_{n=1}^{\infty} \frac{P_n(x)}{(2n-1)(2n+3)}.$$

## 4.4 The Generating Function

*A generating function is a clothesline on which we hang up a sequence of numbers for display.*

Herbert Wilf

Recall that a *generating function* for a sequence $(a_n)_{n\in\mathbb{N}}$ is simply the power series

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n.$$

If $G(a_n; x)$ happens to be the Maclaurin series expansion of some function $g$, then $g$ is also called the generating function of $(a_n)$.

In our case, we are after a function $G(x; t)$ whose coefficients in the Maclaurin series expansion are the Legendre polynomials. It can be shown, using complex analysis, that the desired function is $1/\sqrt{1-2xt+t^2}$, i.e.

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n.$$

The proof is available in section 4.2 of [2].

**Exercise 19.** Find the first five Legendre polynomials $P_1(x), \ldots, P_5(x)$ by using Maclaurin's theorem for $f(t) = 1/\sqrt{1-2xt+t^2}$.

## 4.5 Recurrence Relations

The Legendre polynomials satisfy some recurrence relations which can easily be obtained from the generating function. Indeed, differentiating the generating function $G(x; t)$ with respect to $t$ yields

$$\frac{\partial G}{\partial t}(x; t) = \frac{x-t}{(1-2xt+t^2)^{3/2}},$$

or, written differently,

$$(1-2xt+t^2)\frac{\partial G}{\partial t} + (t-x)G = 0.$$

Now, since $G = \sum_{n=0}^{\infty} P_n(x)t^n$, we have $\frac{\partial G}{\partial t} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$. Proceeding in a manner similar to the method of power series, we have

$$(1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1} + (t-x)\sum_{n=0}^{\infty} P_n(x)t^n = 0$$

$$\implies P_1(x) - xP_0(x) + \sum_{n=1}^{\infty}[(n+1)P_{n+1} - (2n+1)xP_n(x) + nP_{n-1}(x)]t^n = 0,$$

comparing the coefficient of $t_0$ gives that $P_1(x) - xP_0(x) = 0$, that is, $P_1(x) = xP_0(x)$, and since $P_0(x) = 1$ we get $P_1(x) = x$, as expected. The general coefficient of $t^n$ for $n \geqslant 1$ simplifies to give the second-order recurrence relation

$$(n+1)P_{n+1} - (2n+1)xP_n(x) + nP_{n-1}(x)$$

for the Legendre polynomials. Thus we may compute the Legendre polynomials using the following formula:

$$P_n(x) = \begin{cases} 1 & \text{if } n = 0 \\ x & \text{if } n = 1 \\ \frac{1}{n}\left[(2n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x)\right] & \text{if } n \geqslant 2. \end{cases} \tag{16}$$

**Exercise 20.** (i) Use equation (16) to obtain $P_5(x)$.

(ii) By differentiating $G(x;t)$ with respect to $x$ instead, show that

$$(1-2tx+t^2)\frac{\partial G}{\partial x} - Gt = 0. \tag{17}$$

(iii) Use equation (17), noting that $G = \sum_{n=0}^{\infty} P_n(x)t^n$, to show that for $n \geqslant 1$,

$$P'_{n+1} - 2xP'_n + P'_{n-1} = P_n, \tag{18}$$

using the power-series method in a similar way used to obtain (16).

(iv) Using equations (16) and (18) or otherwise, show that for $n \geqslant 1$,

$$P'_{n+1} - P'_{n-1} = (2n+1)P_n. \tag{19}$$

(v) Use equation (19) to show that $P'_{n+1}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (2(n-2k)+1)P_{n-2k}(x)$.

## 4.6 Other Properties

The following facts about the Legendre polynomials are often useful, their proofs may be found in [2, 3, 4].

- The Legendre polynomials have definite parity, that is, they are odd or even according to the relation
$$P_n(-x) = (-1)^n P_n(x).$$
- For $n \geqslant 1$, we have
$$\int_{-1}^{1} P_n(x)\,dx = 0,$$
which is easily seen since by orthogonality, we have $\int_{-1}^{1} P_n(x)\,dx = \int_{-1}^{1} 1 \cdot P_n(x)\,dx = \int_{-1}^{1} P_0(x)P_n(x)\,dx = 0$. As a consequence, the mean value of a function $f(x)$ over $[-1,1]$ expressed as a sum of Legendre polynomials is simply the leading Fourier-Legendre coefficient $c_0$.
- (Askey-Gasper Inequality) For any non-negative $N$ and $x \in [-1,\infty)$, we have
$$\sum_{n=0}^{N} P_n(x) \geqslant 0.$$
- (Laplace's Integral) For any non-negative $n$ and $x \in \mathbb{C}$, we have that
$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} (x + \sqrt{1-x^2}\cos\phi)^n\,d\phi.$$
- (Mehler-Dirichlet formula) Applying the integral substitution $x = \cos\theta$ in Laplace's integral and applying Cauchy's integral theorem yields
$$P_n(\cos\theta) = \frac{\sqrt{2}}{\pi}\int_0^{\theta} \frac{\cos\frac{2n+1}{2}\psi}{\sqrt{\cos\psi - \cos\theta}}\,d\psi.$$
- (Asymptotic form) For $\theta \in (0,\pi)$, we have
$$P_n(\cos\theta) = J_0(n\theta) + O(\tfrac{1}{n}) = \sqrt{\frac{2}{\pi n\sin\theta}}\sin\left(\frac{2n+1}{2}\theta + \frac{\pi}{4}\right) + O(\tfrac{1}{n}),$$
(where $J_0$ is the zeroth Bessel function of the first kind). Consequently, for $x \in [-1,1]$ we get $\lim_{n\to\infty} P_n(x) = 0$.

**Exercise 21.** (i) Show that all roots of $P_n(x)$ are real and lie in $(-1,1)$.[6]

(ii) Prove that for $x \in [-1,1]$ and $n \geqslant 1$,

$$(1-x^2)^{1/4}|P_n(x)| < \sqrt{\frac{2}{n\pi}}.$$

A proof of this fact can be seen in [3].

# References

[1] O. Rodrigues, *De l'attraction des sphéroïdes*, Thesis for the Faculty of Science of the University of Paris, 1816.

[2] N. N. Lebedev and Richard R. Silverman, *Special functions and their applications*, Dover Publications, 1972.

[3] G. Szegő, *Orthogonal polynomials*, Providence: American Mathematical Society, 1975.

[4] E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, American Mathematical Society, Volume 38, no. 7, page 479, 1932.

[6] Hint: use Rolle's theorem.