1 Hyperbolic Functions

In this chapter, we introduce the hyperbolic functions and discuss some of their properties.

1.1 Remembering the circular functions

Recall that the trigonometric functions $\cos\theta$ and $\sin\theta$ are defined as the x- and y-coordinates obtained when one travels a distance of θ along the unit circle $x^2 + y^2 = 1$, starting from the point $(1, 0)$. For this reason, it might be better

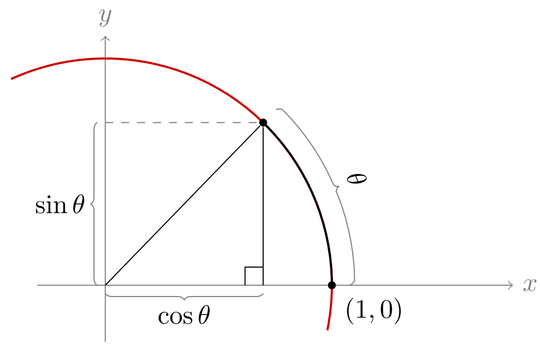

FIGURE 1: The functions $\cos\theta$ and $\sin\theta$

to call these the *circular* functions. In A-level, we saw that using the complex exponential $e^{i\theta} = \cos\theta + i\sin\theta$, we may express

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{1}$$

The hyperbolic functions are analogues of the circular functions, used to study points on the curve $x^2 - y^2 = 1$ (as opposed to $x^2 + y^2 = 1$). This is called a *hyperbola*. Indeed, just as $(\cos\theta, \sin\theta)$ describes the general points on the circle, we have that $(\cosh\theta, \sinh\theta)$ is the point obtained when we travel along the right-half of the curve $x^2 - y^2 = 1$, starting from $(1, 0)$ (see figure 2). But we will not get into the details of this, and instead give our definitions of these functions as follows:

$$\cosh x := \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x := \frac{e^x - e^{-x}}{2}. \tag{2}$$

It shouldn't be too hard to see that these are the analogues of (1) for hyperbolæ.

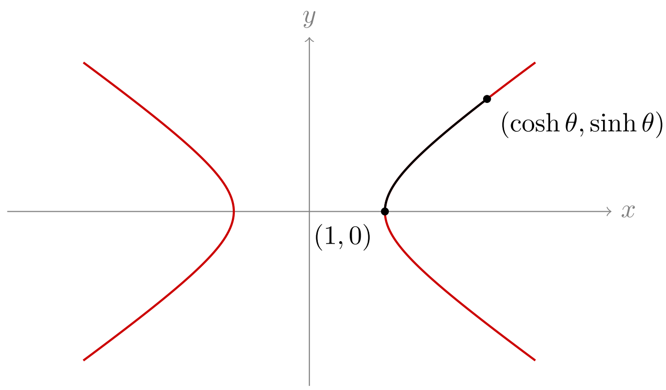

FIGURE 2: The functions $\cosh\theta$ and $\sinh\theta$, depicted as the general x- and y-coordinates of (the right half of) the hyperbola $x^2 - y^2 = 1$

1.2 Osborn's Rule

Osborn's rule is an observation which allows us to translate identities we know to be true about circular functions into identities about the hyperbolic functions. The rule states the following.

> *Any identity in terms of* cos *and* sin *can be translated into one about* cosh *and* sinh *simply by replacing* cos *by* cosh *and* sin *by* sinh*, provided that the sign of any product involving two sines is reversed.*

Examples 1.1. (i) The Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$. This becomes

$$\cosh^2\theta - \sinh^2\theta = 1.$$

(ii) The other Pythageorean identity $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ becomes

$$-\operatorname{cosech}^2\theta = 1 - \coth^2\theta,$$

since $\operatorname{cosec} = \frac{1}{\sin}$ and $\cot = \frac{\cos}{\sin}$, so the corresponding squared terms both contain a product of two sines. (The functions cosech and coth are defined in the next section, but you can probably guess what their definitions are.)

(iii) The double angle identity $\sin 2\theta = 2\sin\theta\cos\theta$ becomes

$$\sinh 2\theta = 2\sinh\theta\cosh\theta,$$

since there is no product of two sines.

(iv) The cosine compound angle identity $\cos(A+B) = \cos A\cos B - \sin A\sin B$ becomes

$$\cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B.$$

You should be able to prove these identities from the definitions. As an example, we prove the double angle identity for sinh (i.e., examples 1.1(iii)):

$$\begin{aligned}\text{RHS} = 2\sinh x\cosh x &= 2\Big(\frac{e^x - e^{-x}}{2}\Big)\Big(\frac{e^x + e^{-x}}{2}\Big)\\ &= 2\Big(\frac{e^{2x} + 1 - 1 - e^{-2x}}{4}\Big)\\ &= \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x = \text{LHS}. \quad \square\end{aligned}$$

Remark 1.2. Observe from (1) and (2) that $\cos ix = \cosh x$ and $\sin ix = i\sinh x$. Thus the occurrence of two sines in an identity introduces a factor of $i^2 = -1$ when replacing the functions with their hyperbolic analogues. This is not a precise proof of Osborn's rule, but this should give you a feel as to why it is true.

1.3 Other Hyperbolic Functions

Just as we do in trigonometry, we introduce four more functions defined in terms of the hyperbolic sine and cosine, mainly because they will make results in calculus look a bit simpler. We have

$$\tanh x := \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

and

$$\operatorname{sech} x := \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \qquad \operatorname{cosech} x := \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}},$$

and finally,

$$\coth x := \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

Example 1.3. Write down the hyperbolic identity which corresponds to the trigonometric identity $1 + \tan^2 x = \sec^2 x$, then prove it from first principles.

The corresponding identity is $1 - \tanh^2 x = \operatorname{sech}^2 x$. Here is the proof:

$$\begin{aligned}\text{LHS} = 1 - \tanh^2 x &= 1 - \Big(\frac{e^x - e^{-x}}{e^x + e^{-x}}\Big)^2\\ &= 1 - \frac{e^{2x} - 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} \qquad \text{(remember this!)}\\ &= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{e^{2x} + 2 + e^{-2x}}\\ &= \frac{4}{e^{2x} + 2 + e^{-2x}} = \Big(\frac{2}{e^x + e^{-x}}\Big)^2 = \sec^2 x = \text{RHS}. \quad \square\end{aligned}$$

1.4 Hyperbolic Equations

Let's solve some equations involving hyperbolic functions. The techniques here are very similar to those used to solve trigonometric equations when appropriate identities can be applied. But we can also just use the definitions.

Example 1.4. Solve $3\sinh x - \cosh x = 1$.

Using the definitions, we have

$$\begin{aligned}
& & 3\sinh x - \cosh x &= 1 & \\
\implies & & 3\Big(\frac{e^x - e^{-x}}{2}\Big) - \Big(\frac{e^x + e^{-x}}{2}\Big) &= 1 & \\
\implies & & 3e^x - 3e^{-x} - e^x - e^{-x} &= 2 & \\
\implies & & 2e^x - 4e^{-x} &= 2 & \\
\implies & & e^x - 1 - 2e^{-x} &= 0 & (\times e^x) \\
\implies & & e^{2x} - e^x - 2 &= 0 & \\
\implies & & (e^x - 2)(e^x + 1) &= 0 & \\
\implies & & e^x = 2 \quad \text{or} \quad e^x &= -1 & \text{(contradiction)} \\
\implies & & e^x &= 2 & \\
\therefore & & x &= \log 2. &
\end{aligned}$$

This technique can work for any equation of the form $a\sinh x + b\cosh x = c$. If it happens that $c = 0$, we can actually simplify the equation since we can divide by $\cosh x$ and get

$$a\sinh x + b\cosh x = 0 \xRightarrow{\div \cosh x} a\tanh x + b = 0 \implies \tanh x = -\tfrac{b}{a}$$

But can we apply $\tanh^{-1}(\,\cdot\,)$ at this stage, just as we would when solving a trigonometric equation? It turns out the answer is yes, and we will discuss the inverses of hyperbolic functions in the next section.

Exercise 1.5. 1. Prove that the point $(\cosh\theta, \sinh\theta)$ does actually lie on the hyperbola

$$x^2 - y^2 = 1$$

since we took our definitions of cosh and sinh to be those in (2).

2. For each of the following trigonometric identities, state the corresponding hyperbolic identity, and prove it from first principles.

   (a) $\sin(x+y)\sin(x-y) = (\sin x + \sin y)(\sin x - \sin y)$

   (b) $\sec^2 x + \operatorname{cosec}^2 x = \sec^2 x \operatorname{cosec}^2 x$

   (c) $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

3. Solve the following equations.

(a) $\cosh x + \sinh x = 1$ (b) $\cosh x + 3\sinh x = 5$

(c) $2e^x \cosh x = 11$ (d) $3\sinh x = e^x + 3$

(e) $\cosh 2x + \sinh 2x = 2$ (f) $\cosh 2x + \sinh x = 2$

(g) $\cosh x - 3\cosh 3x + \cosh 5x = 0$

[HINT: for the last two, don't just substitute the definitions, think of how you would solve them if they were trigonometric equations.]

4. If $\sinh x = \frac{3}{4}$ and $x > 0$, determine the values of $\operatorname{sech} x$, $\tanh x$, $\cosh 2x$ and $\tanh 2x$ without finding x.

5. Simplify the expression $1/(\cosh 3x + \sinh 3x)$, hence determine

$$\int_0^{\log 2} \frac{dx}{\cosh 3x + \sinh 3x}.$$

6. Express $13\cosh 2x + 5\sinh 2x$ in the form $R\cosh(2x + \alpha)$, hence determine the maximum value of

$$\frac{5}{26\cosh 2x + 10\sinh 2x + 9},$$

and the value(s) of x at which the maximum is attained.

7. Solve the set of simultaneous equations

$$3\cosh x - \sinh y = 0$$
$$\cosh y + \sinh x = 3.$$

ANSWERS

3. (a) 0, (b) $\log \frac{1}{4}(5 + \sqrt{33})$, (c) $\frac{1}{2}\log 10$, (d) $\log(3 + 2\sqrt{3})$, (e) $\frac{1}{2}\log 2$, (f) $\log(\sqrt{2} - 1)$, $\log \frac{1}{2}(1 + \sqrt{5})$, (g) $\log \frac{1}{2}(\sqrt{5} \pm 1)$.

4. $\operatorname{sech} x = \frac{4}{5}$, $\tanh x = \frac{3}{5}$, $\cosh 2x = \frac{17}{8}$, $\tanh 2x = \frac{15}{17}$.

5. $\int_0^{\log 2} e^{-3x}\,dx = \frac{7}{24}$.

6. $12\cosh(2x + \log\frac{3}{2})$, max is $\frac{5}{2\cdot 12+9} = \frac{5}{33}$, occurs when $2x + \log\frac{3}{2} = 0$, i.e., when $x = -\frac{1}{2}\log\frac{3}{2}$.

7. $x = \log\frac{1}{2}(\sqrt{5} - 1)$, $y = \log\frac{1}{2}(7 + 3\sqrt{5})$, or
$x = \log\frac{1}{4}(\sqrt{17} - 1)$, $y = \log\frac{1}{4}(13 + 3\sqrt{17})$.

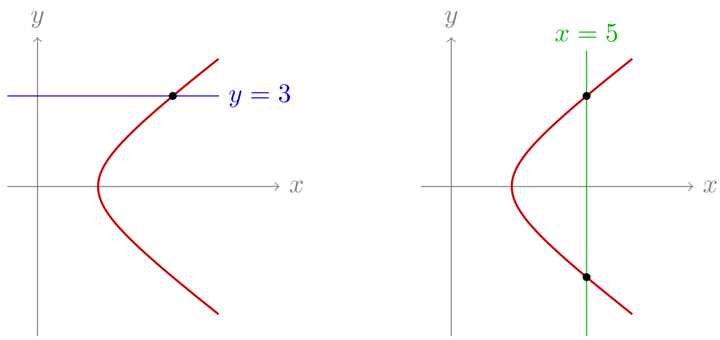

FIGURE 3: Illustration of the solutions of the two equations in example 1.6 on the hyperbola $x^2 - y^2 = 1$.

LESSON 2 *8th December, 2020*

1.5 Inverse Hyperbolic Functions

We can easily solve the equations $x = \cosh y$, $x = \sinh y$ and $x = \tanh y$ to obtain expressions for the inverses of each of the hyperbolic functions.

If we look at figure 2, it is not hard to see that sinh is injective, so it has a well-defined inverse. On the other hand, cosh is not injective, but if we restrict its domain to $[0, \infty)$, we get that $\cosh \restriction [0, \infty)$ is injective, and we call its inverse the *principal value* of $\cosh^{-1}$. In summary, we have

$$\begin{aligned} \sinh^{-1}(x) &:= \log(x + \sqrt{x^2+1}) \\ (\textit{principal value of})\ \cosh^{-1}(x) &:= \log(x + \sqrt{x^2-1}) \quad \text{(defined for } x \geqslant 1) \\ \tanh^{-1}(x) &:= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) \quad \text{(defined for } |x| < 1) \end{aligned}$$

How the relevant domain restrictions come about should be clear from the derivation of these expressions (exercise 1.7.1). You don't have to remember these, they are all given on page 3 of the exam booklet. What's not given is the "other value" of $\cosh^{-1}(x)$, namely $-\log(x + \sqrt{x^2-1})$, which equals $-\cosh^{-1}(x)$.

Example 1.6. Solve the equations $\sinh x = 3$ and $\cosh 2x = 5$.

The solution to the first equation is $x = \sinh^{-1}(3) = \log(3 + \sqrt{10})$.

The second equation becomes $2x = \pm\cosh^{-1}(5) = \pm\log(5 + 2\sqrt{6})$, so the two solutions are $x = \pm\frac{1}{2}\log(5 + 2\sqrt{6})$.

The solutions we obtained here can be visualised on the hyperbola $x^2 - y^2 = 1$, as show in figure 3. What we found are the values of θ such that all points $(\cosh\theta, \sinh\theta)$ have y-coordinate 3 (in the first equation) and x-coordinate 5 (in the second).

1.6 Calculus of Hyperbolic Functions

It is straightforward to verify from the definitions that we have the derivatives

$$\frac{d}{dx}\big(\cosh x\big) = \sinh x \qquad \text{and} \qquad \frac{d}{dx}\big(\sinh x\big) = \cosh x,$$

and consequently, we get the primitives

$$\int \cosh x\, dx = \sinh x + c \qquad \text{and} \qquad \int \sinh x\, dx = \cosh x + c.$$

Similarly, we obtain the following:

- $\dfrac{d}{dx} \tanh x = \operatorname{sech}^2 x$
- $\dfrac{d}{dx} \coth x = -\operatorname{cosech}^2 x$
- $\dfrac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$
- $\dfrac{d}{dx} \operatorname{cosech} x = -\coth x \operatorname{cosech} x$
- $\displaystyle\int \tanh x\, dx = \log(\cosh x) + c$
- $\displaystyle\int \coth x\, dx = \log(\sinh x) + c$
- $\displaystyle\int \operatorname{sech} x\, dx = \tan^{-1}(\sinh x) + c$
- $\displaystyle\int \operatorname{cosech} x\, dx = \log(\tanh \tfrac{x}{2}) + c$

Exercise 1.7.

1. (a) Solve $x = \sinh y$ for y, obtaining an expression for the inverse $\sinh^{-1}(x)$.
   (b) Solve $x = \cosh y$ for y, obtaining two possible expressions for $\cosh^{-1}(x)$. Explain why the condition $x \geqslant 1$ is needed in either case.
   (c) Solve $x = \tanh y$ for y, obtaining an expression for $\tanh^{-1}(x)$. Why do we need $|x| < 1$?
2. Solve the equation

   $$3 \sinh x + 2 \cosh x = 0$$

   without using the definitions.
3. Prove the following (constants of integration omitted).

(a) $\cosh' x = \sinh x$	(b) $\sinh' x = \cosh x$
(c) $\tanh' x = \operatorname{sech}^2 x$	(d) $\coth' x = -\operatorname{cosech}^2 x$
(e) $\operatorname{sech}' x = -\operatorname{sech} x \tanh x$	(f) $\operatorname{cosech}' x = -\operatorname{cosech} x \coth x$
(g) $\int \tanh x\, dx = \log(\cosh x)$	(h) $\int \coth x\, dx = \log(\sinh x)$
(i) $\int \operatorname{sech} x\, dx = \tan^{-1}(\sinh x)$	(j) $\int \operatorname{cosech} x\, dx = \log(\tanh \frac{x}{2})$

4. Show, using appropriate substitutions, that

   (a) $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$ (b) $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \cosh^{-1}\left(\frac{x}{a}\right)$

5. Prove that $y = \tanh x$ is the unique solution to the differential equation

$$y' + y^2 = 1$$

   with $y(0) = 0$.

6. Consider the differential equation

$$\left(\frac{y}{A}\right)^2 - \left(\frac{dy}{dx}\right)^2 = 1.$$

   (a) Prove that any solution satisfies $y(x) > 0$ for all x, or $y(x) < 0$ for all x.

   (b) Hence, show that any solution has the form

$$y(x) = A\cosh\left(\frac{x}{A} + B\right)$$

   where B is a constant.

7. Prove that the area under the curve $y = \cosh x$ for x in the range $a \leqslant x \leqslant b$ is equal to its arclength.

1.7 The Graphs of Hyperbolic Functions

We can reason about what the graphs of the hyperbolic functions look like by their definition. If we think of what happens when we add e^x to e^{-x} and divide the result by 2, we get the graph of $y = \cosh x$, which we can see in figure 4. For the graph of $y = \sinh x$, we instead add e^x to $-e^{-x}$, and then halve the result. This can be seen in figure 5.

Finally for $y = \tanh x$, we can reason about what happens when we divide $\sinh x$ by $\cosh x$. It will be helpful also to note that

$$\tanh x = 1 - \frac{2}{1+e^{2x}},$$

from which we easily see that $-1 < \tanh x < 1$ for all x.

We can deduce some properties about the hyperbolic functions from their graphs. (Most of these we already deduces by other methods, but it's good to list them here and go through them with their graphs in mind):

- sinh, coth and tanh are continuous and differentiable everywhere on their domain.

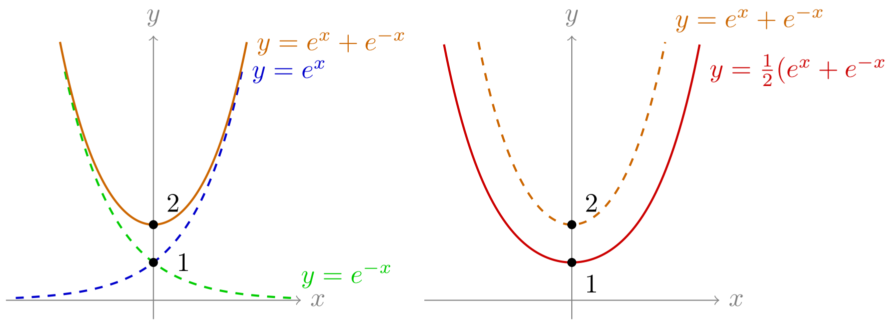

FIGURE 4: We obtain the graph of $e^x + e^{-x}$ visually by adding e^x and e^{-x} pointwise, and then we divide the height at each point by 2 to get the graph of $y = \cosh x$.

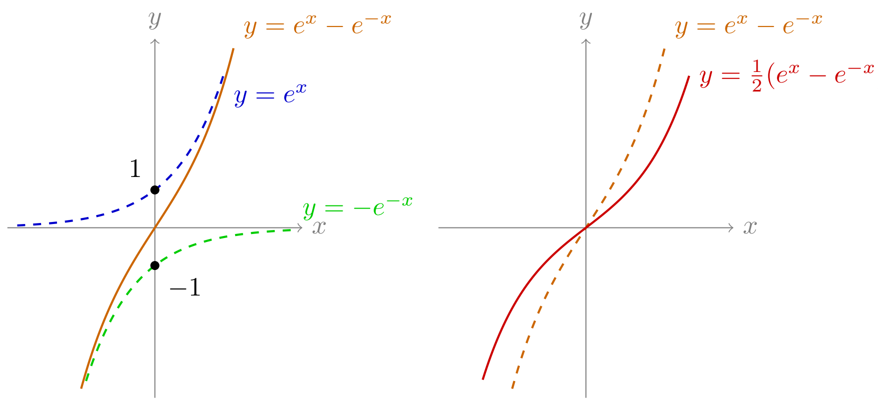

FIGURE 5: We obtain the graph of $e^x - e^{-x}$ visually by adding e^x and $-e^{-x}$ pointwise, and then we divide the height at each point by 2 to get the graph of $y = \sinh x$.

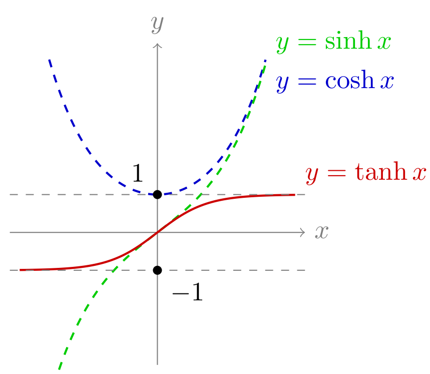

FIGURE 6: We obtain the graph of $\tanh x$ by dividing $\sinh x$ by $\cosh x$ pointwise. Note the asymptotes at $y = \pm 1$.

FIGURE 7: Catenaries in the real world

- $\cosh x \geqslant 1$ for all x.
- $\sinh x \sim \cosh x$ as $x \to \infty$, and $\sinh \sim -\cosh x$ as $x \to -\infty$.
- $\tanh x \sim 1$ as $x \to \infty$, and $\tanh x \sim -1$ as $x \to -\infty$.
- $y = \cosh x$ has one turning point, namely a minimum at $(0, 1)$.
- $y = \sinh x$ has no turning points, it is a strictly increasing function with an oblique inflexion point at $x = 0$.

Note. Take a look at this Wikipedia article if you are unfamiliar with the asymptotic notation $f(x) \sim g(x)$ which we use in some of the points above.

Remark 1.8 (Catenaries). When a chain or wire hangs loose from two positions of equal height, it does so in such a way which minimises gravitational potential energy. The resulting curve they form is called a *catenary* (from the Latin *catena*, meaning *chain*), see figure 7.

Many people (including Galileo) instinctively suspect that catenaries have the shape of a parabolic curves, but actually, their shape is that of a hyperbolic cosine curve. Indeed, suppose a chain or wire hangs from two poles which are at a distance of $2a$ from each other (we can think of the wire as living in the xy-plane between $-a \leqslant x \leqslant a$). If we let $y(x)$ be the height of the wire from the ground for each $-a \leqslant x \leqslant a$, then the total gravitational potential energy associated with the wire is given by

$$E = \rho g \int_{-a}^{a} y(x)\sqrt{1 + y'(x)^2}\, dx,$$

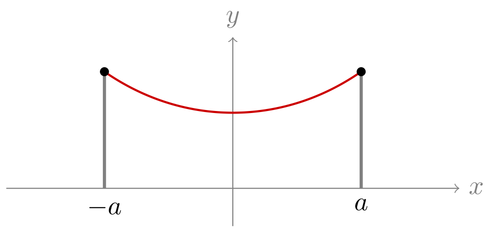

FIGURE 8: Catenary in a coordinate system represented by some function

where ρ is the density of the material that the wire is made of, and g is the acceleration due to gravity (a constant $\approx 9.8\,\mathrm{ms}^{-1}$). The integrand $y\sqrt{1+(y')^2}$ here is just the height of the wire at x (i.e., $y(x)$) multiplied by the arclength over an "infinitesimally small" interval (remember that the arclength of a curve $y(x)$ between $x=a$ and $x=b$ is given by $\int_a^b \sqrt{1+y'(x)^2}\,dx$). Thus, determining a curve which minimises gravitational potential energy is equivalent to finding a function y which minimises the integral. Now, an important theorem from the area known as *calculus of variations* tells us that

$$\int_a^b L(y(x), y'(x))\,dx \text{ is minimal} \qquad \Longleftrightarrow \qquad L(y, y') - y' \,\frac{\partial L(y, y')}{\partial y'} = \text{constant}.$$

where L can be any function (usually called the *Lagrangian* of the problem). In our case, we want that

$$\begin{aligned} & y\sqrt{1+(y')^2} - y' \frac{\partial}{\partial y}\big(y\sqrt{1+(y')^2}\big) = \text{constant} \\ \Longrightarrow \quad & y\sqrt{1+(y')^2} - \frac{y\,(y')^2}{\sqrt{1+(y')^2}} = \text{constant}. \end{aligned}$$

Denoting the constant by A, we can multiply the equation by $\sqrt{1+(y')^2}$ to get

$$\begin{aligned} & y(1+(y')^2) - y\,(y')^2 = A\sqrt{1+(y')^2} \\ \Longrightarrow \quad & y = A\sqrt{1+(y')^2} \\ \Longrightarrow \quad & \left(\frac{y}{A}\right)^2 - (y')^2 = 1. \end{aligned}$$

From exercise 1.7.6(b), we get that the solution to this equation is

$$y(x) = A\cosh\left(\frac{x}{A}\right),$$

where it follows that $B = 0$ since $y(a) = y(-a)$.

An inverted catenary is the most structurally stable of all possible arches, since such an arch redirects the vertical force of gravity into compression forces pressing along the arch's curve. The dome of the Basilica of Our Lady of Mount Carmel in Valletta is an example of a catenary arch (figure 9).