MATSEC SEPTEMBER 2017, PAPER 1, QUESTION 5

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5(a). Express $\cos\theta - \sqrt{3}\sin\theta$ in the form $R\cos(\theta+\alpha)$ where R is a positive number and α is an angle measured in radians between 0 and $\pi/2$.

Hence, sketch the graph of $y = \cos\theta - \sqrt{3}\sin\theta$, given that $0 \leqslant \theta \leqslant 2\pi$, showing clearly the intercepts of the graph with the lines $y = 0, y = \pm 1, y = \pm 2$.

[7 marks]

Solution. We expand $R\cos(\theta+\alpha)$ using the compound angle identity for cosine, so that we may compare coefficients.

$$\begin{aligned}\cos\theta - \sqrt{3}\sin\theta &= R\cos(\theta+\alpha)\\ &= R\cos\alpha\cos\theta - R\sin\alpha\sin\theta,\end{aligned}$$

so we want that

$$\begin{cases} R\cos\alpha = 1 & \text{①}\\ R\sin\alpha = \sqrt{3}. & \text{②}\end{cases}$$

If we do ② ÷ ①, we get $\tan\alpha = \sqrt{3}$, and we may take α to be the principal value $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$. To find R, we take advantage of the Pythagorean identity, noting that $①^2 + ②^2$ gives $R^2 = 4$, i.e., $R = 2$.

Thus we have

$$\cos\theta - \sqrt{3}\sin\theta = 2\cos(\theta + \tfrac{\pi}{3}).$$

Now to sketch the graph of $y = 2\cos(\theta + \frac{\pi}{3})$, we start by drawing the regular cosine curve in the range $0 \leqslant \theta \leqslant 2\pi$ (figure 1). Next, we apply the graphical transformation $f(\theta) \mapsto f(\theta + \frac{\pi}{3})$, which has the effect of translating the graph to the *left* by $\frac{\pi}{3}$ units. If we keep track of the θ-intercepts, $\frac{\pi}{2}$ becomes $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$,

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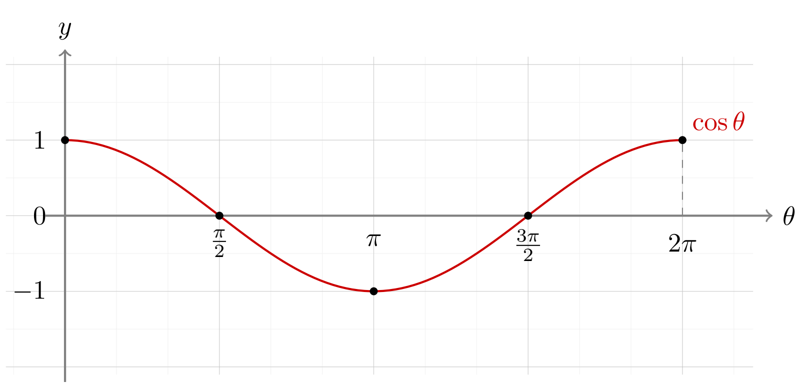

FIGURE 1. Sketch of $y = \cos\theta$

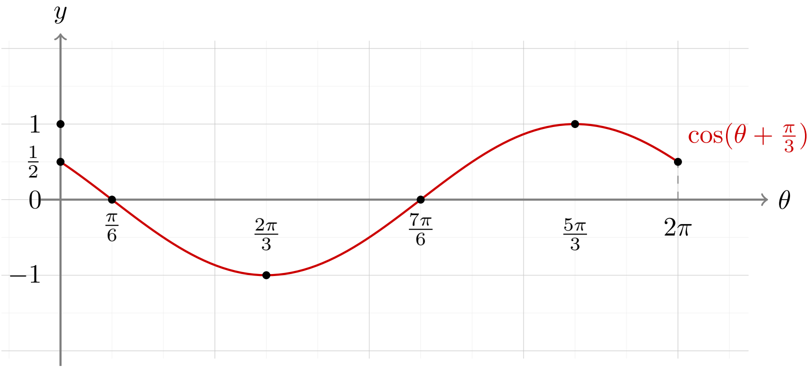

FIGURE 2. Sketch of $y = \cos(\theta + \frac{\pi}{3})$

and $\frac{3\pi}{2}$ becomes $\frac{7\pi}{6}$. Similarly, the trough at π moves to $\frac{2\pi}{3}$, and the peak at 2π moves to $\frac{5\pi}{3}$. The y-intercept is now $\cos(0 + \frac{\pi}{3}) = \frac{1}{2}$ (figure 2). Finally, doing the transformation $f(\theta) \mapsto 2f(\theta)$ to this graph, we obtain the desired graph. This has the effect of a vertical stretch by a factor of two.

Now we've kept track of a lot of information from the original cosine curve, so the intercepts of the curve with the line $y = 0$ (i.e., the θ-axis) and $y = \pm 2$ (i.e., the lines where the peak and trough are located) are all properly labelled. What we still need to locate are the intercepts with $y = \pm 1$. These can easily be found by solving the equations $2\cos(\theta + \frac{\pi}{3}) = \pm 1$. Indeed, for $y = 1$, we have

$$\begin{aligned}
& & 2\cos(\theta + \tfrac{\pi}{3}) &= 1 & \\
&\Longrightarrow & \cos(\theta + \tfrac{\pi}{3}) &= \tfrac{1}{2} & \\
&\Longrightarrow & (\theta + \tfrac{\pi}{3})_{\text{pv}} &= \cos^{-1}(\tfrac{1}{2}) & \\
& & &= \tfrac{\pi}{3} & \\
&\Longrightarrow & \theta + \tfrac{\pi}{3} &= \pm\tfrac{\pi}{3} + 2\pi n & (n \in \mathbb{Z}) \\
&\therefore & \theta &= -\tfrac{\pi}{3} \pm \tfrac{\pi}{3} + 2\pi n & \\
& & &= \tfrac{\pi}{3}(-1 \pm 1 + 6n) & (n \in \mathbb{Z})
\end{aligned}$$

and the values of θ this gives in the range $0 \leqslant \theta \leqslant 2\pi$ are $\theta = 0, \frac{4\pi}{3}, 2\pi$. Similarly, if we solve $2\cos(\theta + \frac{\pi}{3}) = -1$, we obtain the solutions $\theta = \frac{\pi}{3}, \pi$. The graph with all required labels, and therefore the "final answer", can be seen in figure 3. □

5(b). Find the sum of the infinite geometric series

$$1 + \frac{\sec\theta}{1 - \sqrt{3}\tan\theta} + \frac{\sec^2\theta}{(1 - \sqrt{3}\tan\theta)^2} + \frac{\sec^3\theta}{(1 - \sqrt{3}\tan\theta)^3} + \cdots$$

giving your answer in terms of θ. For what values of θ in the interval $[0, 2\pi]$ is your answer valid?

[3 marks]

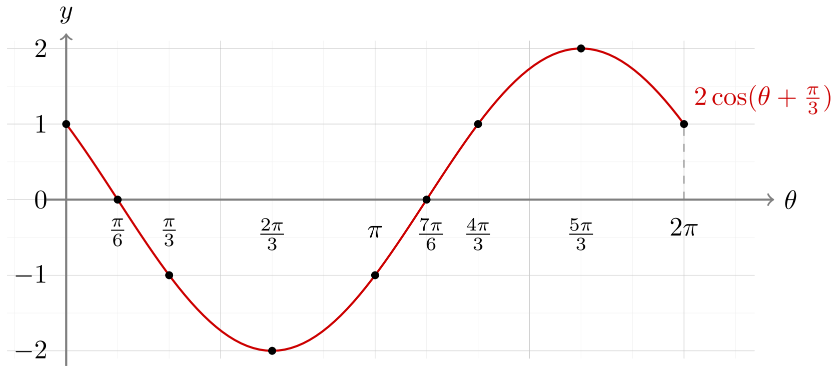

FIGURE 3. Sketch of $y = 2\cos(\theta + \frac{\pi}{3})$

Solution. This is a geometric series with first term 1, so in other words, it's just $1 + r + r^2 + \cdots$, where r is the common ratio. Such series have infinite sum

$$\frac{1}{1-r} = \frac{1}{1 - \frac{\sec\theta}{1-\sqrt{3}\tan\theta}}$$

$$= \frac{1}{\frac{1-\sqrt{3}\tan\theta - \sec\theta}{1-\sqrt{3}\tan\theta}}$$

$$\therefore \qquad S_\infty = \frac{1-\sqrt{3}\tan\theta}{1-\sqrt{3}\tan\theta - \sec\theta}.$$

Now this converges when the common ratio r satisfies $|r| < 1$, i.e., when

$$\left|\frac{\sec\theta}{1-\sqrt{3}\tan\theta}\right| < 1.$$

The denominator here has suspicious coefficients (1 and $\sqrt{3}$), very similar to the function we had to deal with in part (a). In fact, if we multiply the numerator and denominator of the fraction by $\cos\theta$, we get

$$\left|\frac{1}{\cos\theta - \sqrt{3}\sin\theta}\right| < 1,$$

which is precisely the reciprocal of our function. So now the question is, how can we use what we did in part (a) to help us solve this inequality for θ? Well, it's not that hard actually, we simply need to note that $\frac{1}{\text{something}}$ is smaller than 1 (in size) when the denominator is larger than 1 (in size). With a quick glance at figure 3, which is a plot of the denominator, we see that it is larger than 1 (in size) for θ in $\frac{\pi}{3} < \theta < \pi$, and again in $\frac{4\pi}{3} < \theta < 2\pi$. Thus the formula

$$1 + \frac{\sec\theta}{1-\sqrt{3}\tan\theta} + \frac{\sec^2\theta}{(1-\sqrt{3}\tan\theta)^2} + \frac{\sec^3\theta}{(1-\sqrt{3}\tan\theta)^3} + \cdots = \frac{1-\sqrt{3}\tan\theta}{1-\sqrt{3}\tan\theta-\sec\theta}$$

we obtained is valid for $\theta \in (\frac{\pi}{3}, \pi) \cup (\frac{4\pi}{3}, 2\pi)$. □