Mock Assessment Test

Luke's Maths Lessons*
Ħal Tarxien, Malta

Advanced Level March 2022

Instructions

The goal of this test is to prepare you for your MATSEC advanced level pure mathematics exam. The topics assessed here are those which usually fall under the label *precalculus*; namely, basic algebra, coordinate geometry, functions, inequalities and trigonometry; as well as those which form part of *calculus*, namely, differentiation and integration.

Read the following instructions carefully.

- This test consists of **6 questions** and carries **60 marks**.
- You have **2 hours** to complete this test.
- Attempt **all** questions.

*https://maths.mt

1. (a) If $y = \sqrt{x}e^{x^2}$, show that

$$4x^2\frac{d^2y}{dx^2} - 8x\frac{dy}{dx} + (5 - 16x^4)y = 0.$$

(b) A curve is given by the implicit equation $(x^2 + y^2 - 1)^3 = x^2y^3$. Find the equation of the normal through $(1, 1)$.

[5, 5 marks]

2. (a) Use partial fractions to find

$$\int \frac{2x+1}{x^3-1}\,dx.$$

(b) Find

$$\int_1^e \frac{1+\log(x^2)}{x}\,dx.$$

[7, 3 marks]

3. Consider the general equation $ax + by + c = 0$ of a line ℓ, and a point $A = (h, k)$; and assume that $b \neq 0$.

(a) Show that any point on ℓ can be written in the form $P_t = \left(t, -\frac{c+at}{b}\right)$ for an appropriate value of t.

(b) Show that the distance between the general point P_t on ℓ and the point A is given by

$$\sqrt{(h-t)^2 + \left(k + \tfrac{c+at}{b}\right)^2}.$$

Hence, by differentiation with respect to t, show that this expression is minimised when $t = \frac{b^2h-abk-ac}{a^2+b^2}$.

(c) Deduce that the (shortest) distance from the point (h, k) to the line $ax + by + c = 0$ is given by

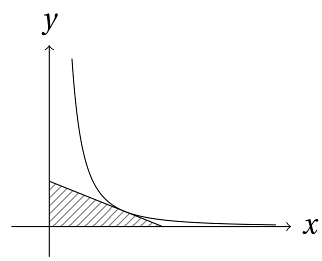
$$\frac{|ah+bk+c|}{\sqrt{a^2+b^2}}.$$

[2, 5, 3 marks]

4. (a) Show that for all x and y,

$$\sin(x+y)\sin(x-y) = (\sin x + \sin y)(\sin x - \sin y).$$

(b) Find the equation of the tangent to $y = \frac{1}{x^2}$, given that the shaded area is $9/4$.



[4, 6 marks]

5. The functions f and g are given by

$$f(x) = 2x^2 + 1 \qquad \text{and} \qquad g(x) = \sqrt{x-3}.$$

(a) State the domain and range of the functions f and g.

(b) Restrict the domain of f so that it becomes an injection. For this restricted domain, determine an expression for f^{-1}.

(c) Determine an expression for $(f \circ g)(x)$.

(d) Carefully state the domain and range of $f \circ g$.

[2, 3, 1, 4 marks]

6. (a) Solve the equation $\log_2 x + \log x = a \log_x 2$, where a is a constant. [*Note:* $\log \equiv \log_e$.]

(b) Find all integers n for which the quadratic equation

$$x^2 + nx + 1 = x$$

has no real solutions.

[5, 5 marks]

Answers

Here are some hints and answers to the test questions (these are *not* detailed solutions: if you present your answers the same way I do here, you will not get marks since there is virtually no working for most of them).

1. (a) *Hint:* Just work out dy/dx and d^2y/dx^2, and plug everything into the LHS; everything should simplify and you get zero.

   (b) *Hint:* Differentiate both sides with respect to x, using the chain rule on the LHS to avoid having to cube stuff.

   The derivative is $\frac{dy}{dx} = \frac{2xy^3-6x(x^2+y^2-1)^2}{-3x^2y^2+6y(x^2+y^2-1)^2}$.

   The normal is $4y = 3x+1$.

2. (a) Using the booklet formula for a^3-b^3 (or the factor theorem with the obvious factor $(x-1)$), the denominator becomes $(x-1)(x^2+x+1)$, and by partial fractions, the integral becomes

$$\begin{aligned}
&\int \frac{dx}{x-1} - \int \frac{x}{x^2+x+1}\,dx \\
&= \log(x-1) - \frac{1}{2}\int \frac{2x}{x^2+x+1}\,dx \\
&= \log(x-1) - \frac{1}{2}\left(\int \frac{2x+1}{x^2+x+1}\,dx - \int \frac{dx}{x^2+x+1}\right) \\
&= \log(x-1) - \frac{1}{2}\log(x^2+x+1) + \frac{1}{2}\int \frac{dx}{x^2+x+1} \\
&= \log\left(\frac{x-1}{\sqrt{x^2+x+1}}\right) + \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \log\left(\frac{x-1}{\sqrt{x^2+x+1}}\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c.
\end{aligned}$$

   (b) Observe that the integral is

$$\begin{aligned}
\int_1^e (1+2\log x)\cdot\tfrac{1}{x}\,dx &= \int_1^e (1+2\log x)\,d(\log x) \\
&= \left[\log x + \log^2 x\right]_1^e = 2.
\end{aligned}$$

   (Equivalently, put $u = \log x$.)

3. (a) *Hint:* If (x, y) lies on the line ℓ, it must satisfy the equation, i.e., the y-coordinate is related to the x-coordinate by $ax + by + c = 0$. Since $b \neq 0$, we can divide by b to obtain the y-coordinate for any given x coordinate: $y = -\frac{ax+c}{b}$. Thus every point P on ℓ has the form $\left(t, -\frac{at+c}{b}\right)$, where t is the x-coordinate of P.

   (b) *Hint:* Just use the distance formula for the distance between two points.

   For the differentiation part, use the chain rule and set the derivative equal to zero to find the t which minimises the expression.

   (c) *Hint:* Plug in the value of t which minimises the distance into the distance formula itself. If you simplify, you should get

$$\sqrt{\frac{(ah+bk+c)^2}{a^2+b^2}} = \frac{|ah+bk+ck|}{\sqrt{a^2+b^2}}.$$

4. (a) *Hint:* You can start from either side, but LHS is probably easier; expand using the compound angle identities.

   (b) *Hint:* Assuming we have a tangent at the point $\left(a, \frac{1}{a^2}\right)$, we can find its equation in general (using differentiation), and determine the area of the triangle, whose side lengths are defined by the intercepts of the tangent.

   The tangent will have equation $a^3 y + 2x = 3a$, so the intercepts with the coordinate axes are $x = \frac{3}{2}a$ and $y = \frac{3}{a^2}$. Thus the area of the triangle is $\frac{1}{2}bh = \frac{1}{2}(\frac{3}{2}a)(\frac{3}{a^2}) = \frac{9}{4a}$, so it's clear that $a = 1$.

   Hence the tangent is $y + 2x = 3$.

5. (a) Domain of f is $\mathbb{R}$, range of f is $[1, \infty)$
   Domain of g is $[3, \infty)$, range of g is $[0, \infty)$.

   (b) If we restrict the domain of f to $[0, \infty)$, the inverse is $f^{-1}(x) = \sqrt{\frac{x-1}{2}}$.

   (c) $(f \circ g)(x) = 2x - 5$.

   (d) *Hint:* Think of $f \circ g$ as a "machine" with components f and g, in order to find the domain and range.

   The domain of $f \circ g$ is $[3, \infty]$, the range is $[1, \infty]$.

6. (a) *Hint:* Use the change of base formula, the equation becomes

$$\frac{\log x}{\log 2} + \log x = \frac{a \log 2}{\log x}.$$

Multiplying through by $\log x$, we get

$$\left(1 + \frac{1}{\log 2}\right) \log^2 x = a \log 2,$$

and dividing by the coefficient of $\log^2 x$,

$$\log^2 x = \frac{a \log^2 2}{1 + \log 2}.$$

Taking square roots,

$$\log x = (\log 2) \left(\pm \sqrt{\frac{a}{1 + \log 2}} \right),$$

and taking exponentials,

$$x = e^{(\log 2)\left(\pm\sqrt{a/(1+\log 2)}\right)}$$

thus the two solutions are $x = 2^{\pm\sqrt{a/(1+\log 2)}}$.

(b) *Hint:* If we look at the discriminant of $x^2 + (n-1)x + 1$, we get the inequality $(n-1)^2 - 4(1)(1) < 0$, which is satisfied for $-1 < n < 3$.

Thus the only integers satisfying this are $0, 1$ and 2.