# Mock Assessment Test

LUKE'S MATHS LESSONS*
Ħal Tarxien, Malta

*Advanced Level* March 2022

## Instructions

The goal of this test is to prepare you for your MATSEC advanced level pure mathematics exam. The topics assessed here are those which usually fall under the label *precalculus*; namely, basic algebra, coordinate geometry, functions, inequalities and trigonometry; as well as those which form part of *calculus*, namely, differentiation and integration.

Read the following instructions carefully.

- This test consists of **6 questions** and carries **60 marks**.
- You have **2 hours** to complete this test.
- Attempt **all** questions.

*https://maths.mt

1. (a) Use partial fractions to find the integral

$$\int \frac{dx}{(x+1)(x+2)^2}.$$

(b) Determine

$$\int_0^{\pi/2} e^x \sin(2x)\, dx$$

using integration by parts. **[5, 5 marks]**

2. (a) Let $y = \log\left(\frac{1}{\sqrt{\cos(x^2)}}\right)$. Show that

$$\frac{dy}{dx} = x\tan(x^2) \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{4x^2 + \sin(2x^2)}{2\cos^2(x^2)}.$$

(b) A curve is given parametrically by the equations

$$x = t^3 - 3t \qquad \text{and} \qquad y = t^2 + t,$$

where $t \in \mathbb{R}$.

(i) Determine the first derivative $\frac{dy}{dx}$ in terms of $t$.

(ii) Show that the tangent to the curve at the point where $t = \frac{1}{2}$ is

$$32x + 36y + 17 = 0.$$

(iii) Determine the coordinates of the other point on the curve where the tangent has the same gradient as the tangent above.

**[4, 6 marks]**

3. (a) Determine $p$ and $q$, given that $x^4 - 4x^3 + 16x^2 + px + q$ is the square of a quadratic expression.

(b) Solve the inequality

$$\frac{\cos\vartheta + 3}{2\sin\vartheta + 1} \leqslant \frac{1}{2}$$

for $0 \leqslant \vartheta \leqslant 2\pi$.

[*Hint:* Think of how you would solve $\frac{x+3}{2x+1} \leqslant \frac{1}{2}$.] **[4, 6 marks]**

4. A function $f$ is defined for all real values of $x$ by

$$f(x) = 3 - |2x - 1|.$$

(a) Sketch the graph of $y = f(x)$, indicating clearly the points where the graph intersects the coordinate axes. Hence show that the equation $f(x) = 4$ has no solutions.

(b) Determine a simplified expression for $(g \circ f)(x)$, where $g\colon \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = x(x-6)$. If $(g \circ f)(x) = k$ has repeated roots, show that $k = -9$.

(c) By finding the values of $x$ for which $f(x) = x$, solve the inequality $f(x) > x$.

**[4, 3, 3 marks]**

5. (a) A point $P$ is twice as far from the point $A = (3, 2)$ as it is from the point $B = (-1, 5)$.

(i) Determine the equation describing the locus of $P$.

(ii) Where do the line $AB$ and the locus of $P$ intersect?

(b) Express $f(\vartheta) = 2\cos 3\vartheta - 2\sin 3\vartheta$ in the form $R\cos(3\vartheta + \alpha)$, where $R > 0$ and $\alpha \in [0, \frac{\pi}{2}]$. Hence, find the general solution to the equation $f(\vartheta) = 1$, giving your solution in exact form.

**[5, 5 marks]**

6. (a) Determine the real numbers $x$ and $y$ if

$$\log_5(4xy + 1) = 2^{xy-1} - 10y = 2.$$

(b) The roots of $x^2 - 3x + 5 = 0$ are $\alpha$ and $\beta$. Find a quadratic equation whose roots are $1/(\alpha^2 + k)$ and $1/(\beta^2 + k)$.

**[5, 5 marks]**

## Answers

1. (a) The partial fraction expansion should have the shape

$$\frac{1}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2},$$

and clearing denominators, we have

$$1 = A(x+2)^2 + B(x+1)(x+2) + C(x+1).$$

To find the constants, we substitute different values for $x$.

$$\begin{aligned}
x &= -1 \quad \implies A = 1 \\
x &= -2 \quad \implies 1 = C(-1) \implies C = -1 \\
x &= 0 \quad \implies 1 = 4A + 2B + C \implies 1 = 4(1) + 2B + (-1) \implies B = -1,
\end{aligned}$$

and so the integral becomes

$$\begin{aligned}
\int \frac{dx}{(x+1)(x+2)^2} &= \int \left(\frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{(x+2)^2}\right) dx \\
&= \log(x+1) - \log(x+2) - \int (x+2)^{-2}\, dx \\
&= \mathbf{log}\left(\frac{\boldsymbol{x+1}}{\boldsymbol{x+2}}\right) + \frac{\mathbf{1}}{\boldsymbol{x+2}} + \boldsymbol{c}.
\end{aligned}$$

(b) Calling the integral $I$, we have

$$\begin{aligned}
I &= \int_0^{\pi/2} e^x \sin(2x)\, dx \\
&= \int_0^{\pi/2} \sin(2x)\, d(e^x) \\
&= \Big[e^x \sin(2x)\Big]_0^{\pi/2} - \int_0^{\pi/2} e^x\, d(\sin(2x)) \\
&= -2\int_0^{\pi/2} e^x \cos(2x)\, dx \\
&= -2\int_0^{\pi/2} \cos(2x)\, d(e^x) \\
&= -2\left(\Big[e^x \cos(2x)\Big]_0^{\pi/2} - \int_0^{\pi/2} e^x\, d(\cos(2x))\right) \\
&= -2(-e^{\pi/2} - 1 + 2I),
\end{aligned}$$

and so we see that the integral satisfies the equation

$$I = -2(-e^{\pi/2} - 1 + 2I),$$

which we can easily solve to obtain $\boldsymbol{I = \frac{2}{5}(1 + e^{\pi/2})}$.

2. (a) Notice that by laws of logarithms, $y = -\frac{1}{2}\log(\cos(x^2))$. Thus, using the chain rule,

$$\frac{dy}{dx} = -\frac{1}{2} \cdot \frac{1}{\cos(x^2)} \cdot (-\sin(x^2)) \cdot 2x = x\tan(x^2).$$

Next, by the product rule, we have

$$\begin{aligned}\frac{d^2y}{dx^2} &= 1 \cdot \tan(x^2) + x \cdot \sec^2(x^2) \cdot 2x \\ &= \frac{\sin(x^2)}{\cos(x^2)} + \frac{2x^2}{\cos^2(x^2)} \\ &= \frac{\sin(x^2)\cos(x^2) + 2x^2}{\cos^2(x^2)} \\ &= \frac{2\sin(x^2)\cos(x^2) + 4x^2}{2\cos^2(x^2)} \\ &= \frac{\sin(2x^2) + 4x^2}{2\cos^2(x^2)},\end{aligned}$$

where in the last step we invoke the identity $2\sin A\cos A = \sin(2A)$.

(b) (i) We have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \boldsymbol{\frac{2t+1}{3t^2-3}}.$$

(ii) When $t = \frac{1}{2}$, we have $x\left(\frac{1}{2}\right) = -\frac{11}{8}$, $y\left(\frac{1}{2}\right) = \frac{3}{4}$, and $\left.\frac{dy}{dx}\right|_{t=1/2} = -\frac{8}{9}$, and so the equation of the tangent at $\left(-\frac{11}{8}, \frac{3}{4}\right)$ is

$$y - \tfrac{3}{4} = -\tfrac{8}{9}\left(x + \tfrac{11}{8}\right),$$

which simplifies to $32x + 36y + 17 = 0$.

(iii) The gradient of the tangent line in (ii) is $-\frac{8}{9}$, so we want to find the other point on the curve whose corresponding tangent line also has this gradient. This is equivalent to finding the other value of $t$ which gives us $\frac{dy}{dt} = -\frac{8}{9}$, i.e., solving

$$\frac{2t+1}{3t^2-3} = -\frac{8}{9}.$$

This is a quadratic equation with roots $t = \frac{1}{2}$ and $t = -\frac{5}{4}$, i.e., the other point is the point with $t = -\frac{5}{4}$.

When $t = -\frac{5}{4}$, $x\big(-\frac{5}{4}\big) = \frac{115}{64}$ and $y\big(-\frac{5}{4}\big) = \frac{5}{16}$, so the desired pair of coordinates is $\boldsymbol{\big(\frac{115}{64}, \frac{5}{16}\big)}$.

3. (a) Since the coefficient of $x^4$ is 1 (i.e., the polynomial is *monic*), then the quadratic factor must also be monic, i.e., we can assume it equals $x^2+bx+c$ for appropriate $b$ and $c$. Thus we want to find $b$ and $c$ such that

$$\begin{aligned} x^4 - 4x^3 + 16x^2 + px + q &= (x^2+bx+c)^2 \\ &= x^4 + 2bx^3 + (b^2+2c)x^2 + 2bcx + c^2. \end{aligned}$$

Comparing coefficients of $x^3$ we see that we must have $2b = -4$, i.e., $b = -2$, and comparing those of $x^2$, we must have $b^2 + 2c = 16$, so $(-2)^2 + 2c = 16$, which gives $c = 6$.

Thus the polynomial we have is $(x^2 - 2x + 6)^2$.

Now $p$ is the coefficient of $x$, which we can see from the expanded form should be $2bc = 2(-2)(6) = -24$, and similarly we must have $q = c^2 = 36$.

Thus $\boldsymbol{p = -24, q = 36}$.

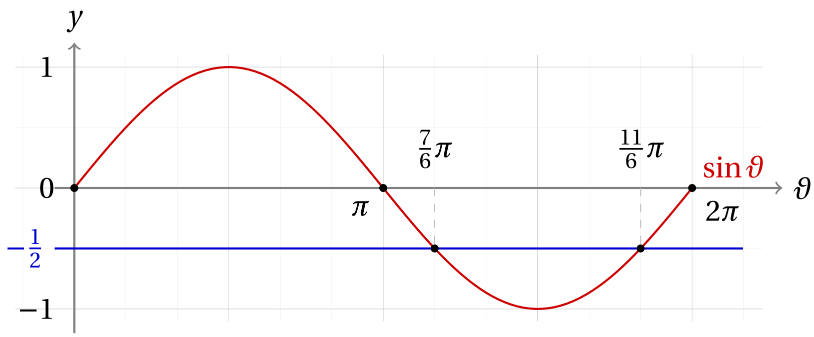
(b) As usual with rational inequalities, since we don't know the sign of the denominator $2\sin\vartheta + 1$, we cannot just multiply both sides by it, but we *can* multiply throughout by its square, which is definitely non-negative. This transforms the inequality into

$$\begin{aligned} & 2(\cos\vartheta+3)(2\sin\vartheta+1) \leqslant (2\sin\vartheta+1)^2 \\ \iff\quad & 2(\cos\vartheta+3)(2\sin\vartheta+1) - (2\sin\vartheta+1)^2 \leqslant 0 \\ \iff\quad & (2\sin\vartheta+1)\big[2\cos\vartheta + 6 - (2\sin\vartheta+1)\big] \leqslant 0 \\ \iff\quad & (2\sin\vartheta+1)(2\cos\vartheta - 2\sin\vartheta + 5) \leqslant 0. \end{aligned}$$

Now $2\cos\vartheta - 2\sin\vartheta + 5$ is at least 1 for any value of $\vartheta$,[†] so it won't have any bearing on whether or not the expression on the LHS is $\leqslant 0$. Thus, we can just divide throughout by it, and we see that the inequality is equivalent to

$$2\sin\vartheta + 1 \leqslant 0,$$

i.e., $\sin\vartheta \leqslant -\frac{1}{2}$. Solving the equation $\sin\vartheta = -\frac{1}{2}$ for $0 \leqslant \vartheta \leqslant 2\pi$, we get the solutions $\frac{7}{6}\pi, \frac{11}{6}\pi$. Thus, with reference to a quick sketch of $y = \sin\vartheta$ in this range,




we see that $\sin\vartheta \leqslant -\frac{1}{2}$ for $\theta$ in the range $\frac{7}{6}\pi \leqslant \vartheta \leqslant \frac{11}{6}\pi$.

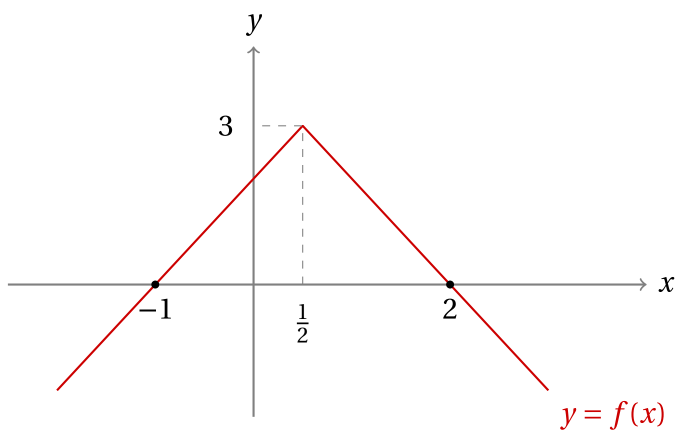
Observe the endpoints of this range make the denominator of our original inequality zero however, so we must exclude them: thus the final answer is $\boldsymbol{\frac{7}{6}\pi < \vartheta < \frac{11}{6}\pi}$.

4. (a) By reasoning graphically about the sequence

$$|x| \quad\longmapsto\quad |x-1| \quad\longmapsto\quad |2x-1| \quad\longmapsto\quad -|2x-1| \quad\longmapsto\quad -|2x-1|+3$$

of affine transformations of $|x|$, we can carry out the corresponding transformations on the graph of $y = |x|$ to end up with the graph of $y = f(x)$. (Note that the order of steps in the sequence is not unique: other sequences are possible.)

[†] This bound is not sharp, but it's easy to obtain since the minimum values of sin and cos are both $-1$. A sharp lower-bound would be $-R+5$, where $R > 0$ is the value needed to express $2\cos\vartheta - 2\sin\vartheta + 5$ in the form $R\cos(\vartheta + \alpha) + 5$.




Since the graph exists for $y \leqslant 3$, then clearly it doesn't intersect the line $y = 4$, consequently the equation $f(x) = 4$ has no solutions.

(b) Noticing that $x(x-6) = (x-3)^2 - 9$ (by completing the square), the working is quite simple:

$$\begin{aligned} (g \circ f)(x) &= (f(x) - 3)^2 - 9 \\ &= (-|2x-1|)^2 - 9 \\ &= 4x^2 - 4x + 1 - 9 \\ \therefore \qquad (g \circ f)(x) &= \mathbf{4(x^2 - x - 2)}. \end{aligned}$$

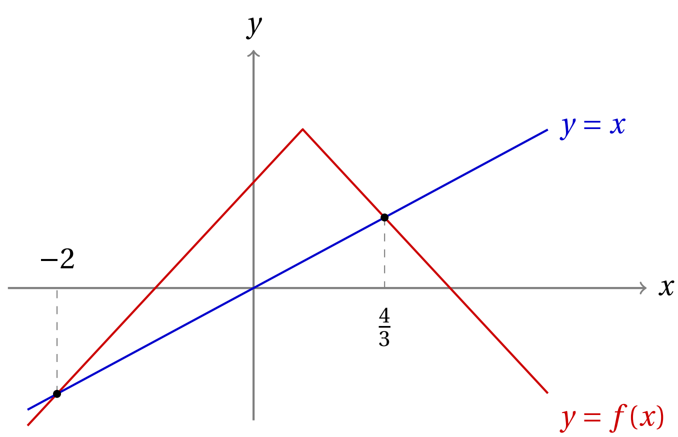
Thus, $(g \circ f)(x) = k$ having repeated roots is equivalent to saying that $4x^2 - 4x + (-8-k) = 0$ has repeated roots. This happens precisely when the quadratic discriminant $\Delta = (-4)^2 - 4(4)(-8-k)$ is zero, i.e., when $16 - 128 - 16k = 0$, i.e., when $k = -9$.

(c) To solve an equation involving moduli, simply remember that $|\text{LHS}| =$

RHS is true if and only if LHS = ±RHS is true.[‡] Thus,

$$\begin{aligned}
&& f(x) &= x \\
\Longrightarrow && 3-|2x-1| &= x \\
\Longrightarrow && 3-x &= |2x-1| \\
\Longrightarrow && 3-x &= \pm(2x-1) \\
\Longrightarrow && 3-x &= \pm 2x \mp 1 \\
\Longrightarrow && x \pm 2x &= 3 \pm 1 \\
\Longrightarrow && 3x = 4 \quad &\text{OR} \quad -x = 2 \\
\Longrightarrow && \boldsymbol{x = \tfrac{4}{3}} \quad &\text{OR} \quad \boldsymbol{x = -2}.
\end{aligned}$$

Now to solve $f(x) > x$, we can make reference to a quick sketch of $y = x$ superimposed on $y = f(x)$:




We see that $f(x) > x$ for $\boldsymbol{-2 < x < \tfrac{4}{3}}$.

5. (a) (i) $P$ satisfies $d(P, A) = 2d(P, B)$, i.e.,

$$\sqrt{(x-3)^2 + (y-2)^2} = 2\sqrt{(x+1)^2 + (y-5)^2}.$$

Squaring both sides, the equation simplifies to

$$3x^2 + 3y^2 + 14x - 36y + 91 = 0,$$

[‡] Alternatively, squaring both sides of something like |LHS| = RHS, we can do away with the moduli.

and completing the square in $x$ and $y$, we obtain

$$\boldsymbol{\left(x+\tfrac{7}{3}\right)^2+(y-6)^2=\tfrac{100}{9}},$$

so the locus of $P$ is a circle centred at $\left(-\frac{7}{3},6\right)$, with radius $\frac{10}{3}$.

(ii) The line through $AB$ has gradient $\frac{5-2}{-1-3}=-\frac{3}{4}$, so its equation is

$$y-2=-\tfrac{3}{4}(x-3),$$

which simplifies to $3x+4y=17$.

To find the points of intersection, we can solve the equations of the line and circle simultaneously. The points are $\boldsymbol{\left(\frac{1}{3},4\right)}$ and $\boldsymbol{(-5,8)}$.

(b) We expand $R\cos(3\vartheta+\alpha)$ using the compound angle identity for cosine, so that we may compare coefficients.

$$\begin{aligned} f(\vartheta)=2\cos 3\vartheta-2\sin\vartheta &= R\cos(3\vartheta+\alpha) \\ &= (R\cos\alpha)\cos 3\vartheta-(R\sin\alpha)\sin 3\vartheta, \end{aligned}$$

so we want that

$$\begin{cases} R\cos\alpha=2 & \text{①} \\ R\sin\alpha=2. & \text{②} \end{cases}$$

If we do ②÷①, we get $\tan\alpha=1$, so we may take $\alpha$ to be the principal value $\tan^{-1}(1)=\frac{\pi}{4}$.

To find $R$, we take advantage of the Pythagorean identity, noting that ①$^2$ + ②$^2$ gives $R^2=8$, i.e., $R=2\sqrt{2}$.

Thus $f(\vartheta)=2\sqrt{2}\cos\left(3\vartheta+\frac{\pi}{4}\right)$, and so the equation we need to solve is

$$\begin{aligned} & 2\sqrt{2}\cos\left(3\vartheta+\tfrac{\pi}{4}\right)=1 \\ \Longrightarrow\quad & \cos\left(3\vartheta+\tfrac{\pi}{4}\right)=\tfrac{\sqrt{2}}{4} \\ \Longrightarrow\quad & \left(3\vartheta+\tfrac{\pi}{4}\right)_{\text{p.v.}}=\cos^{-1}\left(\tfrac{\sqrt{2}}{4}\right) \\ \Longrightarrow\quad & 3\vartheta+\tfrac{\pi}{4}=\pm\cos^{-1}\left(\tfrac{\sqrt{2}}{4}\right)+2\pi\mathbb{Z} \\ \therefore\quad & \boldsymbol{\vartheta=-\tfrac{\pi}{12}\pm\tfrac{1}{3}\cos^{-1}\left(\tfrac{\sqrt{2}}{4}\right)+\tfrac{2}{3}\pi\mathbb{Z}} \end{aligned}$$

6. (a) We solve the simultaneous equations

$$\begin{cases} \log_5(4xy+1) = 2 & \text{①} \\ 2^{xy-1} - 10y = 2. & \text{②} \end{cases}$$

One way to solve these is to start from ①, doing $5^{(\cdot)}$ both sides to get

$$4xy + 1 = 25 \implies xy = 6 \implies x = \tfrac{6}{y}. \qquad \text{③}$$

Substituting this in ②, we get

$$2^{\frac{6}{y}\cdot y-1} - 10y = 2 \implies 2^5 - 10y = 2 \implies 10y = 2^5 - 2 \implies y = 3.$$

Then we can find $x$ using ③. Thus $\boldsymbol{x = 2, y = 3}$.

(b) By Viète's formulae, we immediately have that $\alpha + \beta = 3$ and $\alpha\beta = 5$. Now the sum of the new roots is

$$\begin{aligned} \frac{1}{\alpha^2+k} + \frac{1}{\beta^2+k} &= \frac{\alpha^2+\beta^2+2k}{\alpha^2\beta^2 + (\alpha^2+\beta^2)k + k^2} \\ &= \frac{(\alpha+\beta)^2 - 2\alpha\beta + 2k}{(\alpha\beta)^2 + ((\alpha+\beta)^2 - 2\alpha\beta)k + k^2} \\ &= \frac{3^2 - 2(5) + 2k}{5^2 + (3^2 - 2(5))k + k^2} \\ &= \frac{2k-1}{k^2 - k + 25}, \end{aligned}$$

and similarly, it is easy to see that the product of the roots is

$$\left(\frac{1}{\alpha^2+k}\right)\left(\frac{1}{\beta^2+k}\right) = \frac{1}{k^2-k+25}.$$

Thus the "new" equation is $x^2 - (\text{new sum})x + \text{new product} = 0$, which in our case simplifies to $\boldsymbol{(k^2 - k + 25)x^2 - (2k-1)x + 1 = 0}$.