Mock Assessment Test

LUKE'S MATHS LESSONS*
Ħal Tarxien, Malta

Advanced Level January 2022

Instructions

The goal of this test is to prepare you for your MATSEC advanced level pure mathematics exam. The topics assessed here are those which usually fall under the label *precalculus*; namely, basic algebra, coordinate geometry, functions, inequalities and trigonometry.

Read the following instructions carefully.

- This test consists of **5 questions** and carries **50 marks**.
- You have **1½ hours** to complete this test.
- Attempt **all** questions.

*https://maths.mt

1. Consider the polynomial function $f(x) = 3x^3 - 14x^2 + 8x + 3kx - 2k$, where k is a real constant.

   (a) Evaluate $f\left(\frac{2}{3}\right)$, and deduce a factorisation of f.

   (b) For which value(s) of k does the equation $f(x) = 0$ have three real solutions?

   (c) Solve the inequality $f(x) \geqslant 0$ in the case where $k = \frac{20}{9}$. Illustrate your solutions on a sketch of $y = f(x)$.

   (d) Decompose the rational function $\dfrac{x^2}{f(x)}$ into partial fractions in the case where $k = 2$.

[2, 3, 2, 3 marks]

2. (a) What is the domain of the function $f(x) = \dfrac{x}{1 - \sin(x/2)}$?

   (b) Solve the equations

$$\begin{cases} 3^{x+y} = 9^{9x+5} \\ \log_{3x+2}(3x^2 + y) = 2 \end{cases}$$

   simultaneously.

   (c) Determine the values satisfying the inequality $\dfrac{4}{1+x} \leqslant 2x$.

[2, 4, 4 marks]

3. (a) Find the locus of the point P whose distance from the line $y = 2$ is equal to the distance from the point A with coordinates $(1, 3)$.

   (b) Determine the points where this locus intersects the line ℓ_1 with equation $y = 2x + 3$. Sketch both the locus and the line ℓ_1 on the same axes, indicating the points of intersection.

   (c) Find the equation of the line ℓ_2, which is perpendicular to ℓ_1 and passes through the point A.

   (d) Find the equation of the circle $\mathscr{C}$ which touches both ℓ_1 and ℓ_2 from above, and has radius $\frac{6}{\sqrt{5}}$.

[2, 3, 1, 4 marks]

4. (a) (i) Express $f(x) = 3\cos x + \sqrt{3}\sin x$ in the form

$$f(x) = R\cos(x - \alpha)$$

where $R > 0$ and $\alpha \in \left[0, \frac{\pi}{2}\right]$.

(ii) Sketch $y = f(x)$ in the range $[0, 2\pi]$, clearly indicating the curve's amplitude, and the points where it intersects the coordinate axes.

(iii) Find the minimum value of $1/(f(x)+2)$ for $0 \leqslant x \leqslant \frac{\pi}{2}$, and state the value of x at which this minimum occurs.

(b) (i) Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(ii) By letting $x = \cos\theta$, show that the solutions to the equation

$$8x^3 - 6x = 1$$

are $\cos(20°)$, $\cos(100°)$ and $\cos(140°)$.

[6, 4 marks]

5. Consider the functions $f\colon [3, \infty) \to \mathbb{R}$ and $g\colon [0, 10] \to \mathbb{R}$, defined by

$$f(x) = 2 + \sqrt{x - 3} \qquad \text{and} \qquad g(x) = x(x - 4)$$

for all x in their respective domains.

(a) (i) Which of f and g are injective (i.e., one-to-one)? Explain why.

(ii) Which of them are surjective (i.e., onto)?

(iii) The domain of g is restricted to the interval $[k, 10]$. What is the least value of k possible for which the resulting function is an injection (i.e., one-to-one)?

(b) Find an expression for $(g \circ f)(x)$, and carefully state the domain and range of $g \circ f$ as a composite function.

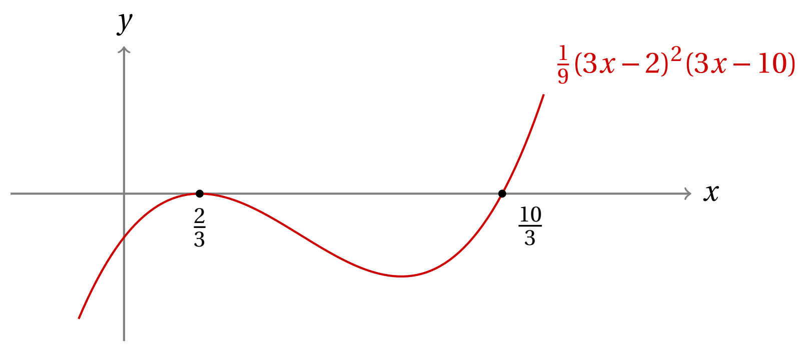
[5, 5 marks]

Solutions

1. (a) $f(\frac{2}{3}) = 0$, so $(3x-2) \mid f$ by the factor theorem. Thus we may write $f(x) = (3x-2)q(x)$ for some polynomial q. By division or otherwise, we can find q and write f as $\boldsymbol{f(x) = (3x-2)(x^2-4x+k)}$.

   (b) The equation $f(x) = 0$ always has $x = \frac{2}{3}$ as a solution. It will have two additional solutions if the quadratic factor has roots. This happens when the discriminant $\Delta = (-4)^2 - 4(1)(k) \geqslant 0$, i.e., when $\boldsymbol{k \leqslant 4}$.

   (c) When $k = \frac{20}{9}$, the problem becomes $\frac{1}{9}(3x-2)^2(3x-10) \geqslant 0$ since the quadratic factors nicely. Thus the curve looks like this:



   and so the solution is $\boldsymbol{x = \frac{2}{3}}$ **or** $\boldsymbol{x \geqslant \frac{10}{3}}$.

   (d) When $k = 2$, the quadratic factor does not split nicely, so we decompose it as

$$\frac{x^2}{(3x-2)(x^2-4x+2)} = \frac{A}{3x-2} + \frac{Bx+C}{x^2-4x+2}.$$

   Taking common denominators and comparing numerators, we have

$$x^2 = A(x^2-4x+2) + (3x-2)(Bx+C).$$

   Putting $x = \frac{2}{3}$, we find that $\frac{4}{9} = -\frac{2}{9}A$, so $A = -2$.

   Next, we can set $x = 0$ to get that $0 = 2A - 2C \implies 0 = 2(-2) - 2C$, so that $C = -2$ also.

   Finally, setting $x = 1$, we get that $1 = -A + B + C \implies 1 = 2 + B - 2$, thus $B = 1$.

   Therefore the decomposition we want is $\boldsymbol{\frac{x-2}{x^2-4x+2} - \frac{2}{3x-2}}$.

2. (a) The expression defining the function f makes sense precisely when the denominator isn't zero. In other words, f is defined for all real values, except those x for which $1 - \sin(\frac{x}{2}) = 0$.

Solving in the usual way, get the principal value

$$\left(\tfrac{x}{2}\right)_{\text{p.v.}} = \sin^{-1}(1) = \tfrac{\pi}{2},$$

and thus the general solution is

$$\tfrac{x}{2} = (-1)^n \cdot \tfrac{\pi}{2} + \pi n \implies x = ((-1)^n + 2n)\pi,$$

where $n \in \mathbb{Z}$. Thus the domain is the set $\mathbb{R}$ excluding all these values, i.e., $\mathbb{R} \setminus \{\boldsymbol{((-1)^n + 2n)\pi : n} \in \mathbb{Z}\}$.

(b) Labelling the equations, we have

$$\begin{cases} 3^{x+y} = 9^{9x+5} & ① \\ \log_{3x+2}(3x^2 + y) = 2. & ② \end{cases}$$

Notice we can rewrite 9 as 3^2 in ① to get

$$3^{x+y} = (3^2)^{9x+5} \implies 3^{x+y} = 3^{2(9x+5)} \implies x + y = 2(9x+5),$$

or, making y subject,

$$y = 17x + 10. \qquad ①'$$

Putting this in ②, we get

$$\begin{aligned} \log_{3x+2}(3x^2 + 17x + 10) = 2 &\implies (3x+2)^2 = 3x^2 + 17x + 10 \\ &\implies 6x^2 - 5x - 6 = 0 \\ &\implies (3x+2)(2x-3) = 0 \\ &\implies x_1 = -\tfrac{2}{3} \text{ or } x_2 = \tfrac{3}{2}. \end{aligned}$$

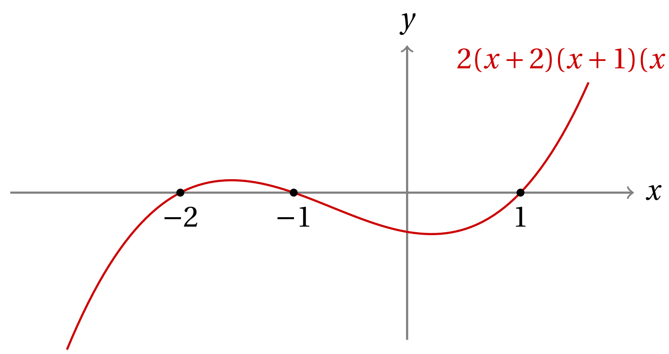
Notice that x_1 does not correspond to a valid solution however, since it gives the logarithm in ② a base of 0. Thus the only possible value for x is $\frac{3}{2}$. Using ①', we can get the corresponding y-value of $\frac{71}{2}$.

Thus the final solution is $\boldsymbol{x = \frac{3}{2}}$ **and** $\boldsymbol{y = \frac{71}{2}}$.

(c) Since we have an inequality, we can only multiply throughout by quantities whose sign we know. (In particular, we CANNOT multiply throughout by $(1+x)$). If we multiply by $(1+x)^2$, which is surely not negative, the inequality becomes

$$\begin{aligned} 4(1+x) \leqslant 2x(1+x)^2 &\implies 2x(1+x)^2 - 4(1+x) \geqslant 0 \\ &\implies (1+x)[2x(1+x)-4] \geqslant 0 \\ &\implies 2(x+2)(x+1)(x-1) \geqslant 0 \end{aligned}$$

Doing a quick sketch:



We see that $f(x)$ is $\geqslant 0$ for $\mathbf{-2 \leqslant x < -1}$ **or** $\boldsymbol{x \geqslant 1}$. Notice we exclude the endpoint $x = -1$ since it corresponds to division by zero in the original rational inequality.

3. (a) Let $P = (x, y)$. The condition we have is $d(P, \ell) = d(P, A)$, where ℓ is the line $y - 2 = 0$ and $A = (1, 3)$. Using the formula in the booklet,

$$\frac{|0x + 1y - 2|}{\sqrt{0^2 + 1^2}} = \sqrt{(x-1)^2 + (y-3)^2} \implies (y-2)^2 = (x-1)^2 + (y-3)^2,$$

and this simplifies to the equation of the parabola $\boldsymbol{y = \frac{1}{2}x^2 - x + 3}$.

(b) For intersection, we solve their equations simultaneously.

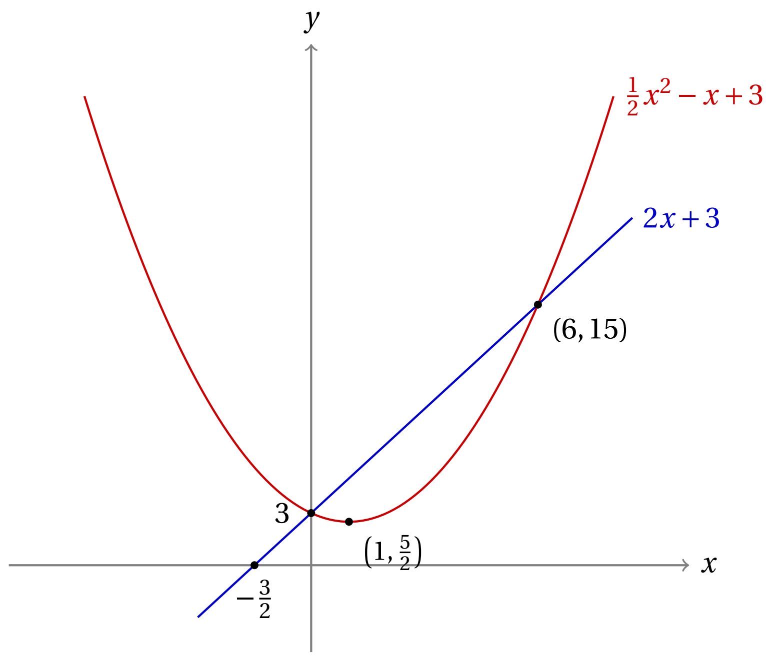
$$\tfrac{1}{2}x^2 - x + 3 = 2x + 3 \implies x(x-6) = 0 \implies x = 0 \text{ or } x = 6.$$

Using the equation of ℓ_1 to find the corresponding y-coordinates, we get the points **(0, 3) and (6, 15)**.

For the sketch of the parabola, notice that $a = \frac{1}{2} > 0$, so we have an upright parabola (i.e., one with a minimum turning point). Also, setting $x = 0$ gives us the y-intercept $y = 3$. If we try to determine the x-intercepts, we realise that there aren't any (the quadratic discriminant $\Delta = -5 < 0$), but if we complete the square, we see that $y = \frac{1}{2}(x-1)^2 + \frac{5}{2}$, so it has minimum turning point $\left(1, \frac{5}{2}\right)$ (this will help us when sketching).

For the line, when we put $x = 0$, we get the y-intercept $y = 3$. Putting $y = 0$, we get $x = -\frac{3}{2}$.

Here's the sketch:



(c) ℓ_1 perpendicular to $\ell_1 \implies m_2 = -1/m_1$, i.e., $m_2 = -\frac{1}{2}$.

Thus $\ell_2\colon\ y - 3 = -\frac{1}{2}(x-1)$, which simplifies to $\boldsymbol{x + 2y = 7}$.

(d) If we add ℓ_2 to the sketch above, we see that what we are looking for is the equation of the circle depicted in **??**.

Let the centre C of the circle be (a, b). Then $d(C, \ell_1) = d(C, \ell_2) = \frac{6}{\sqrt{5}}$, which becomes the equations

$$\frac{|2a - b + 3|}{\sqrt{2^2 + 1^2}} = \frac{|a + 2b - 7|}{\sqrt{1^2 + 2^2}} = \frac{6}{\sqrt{5}}.$$

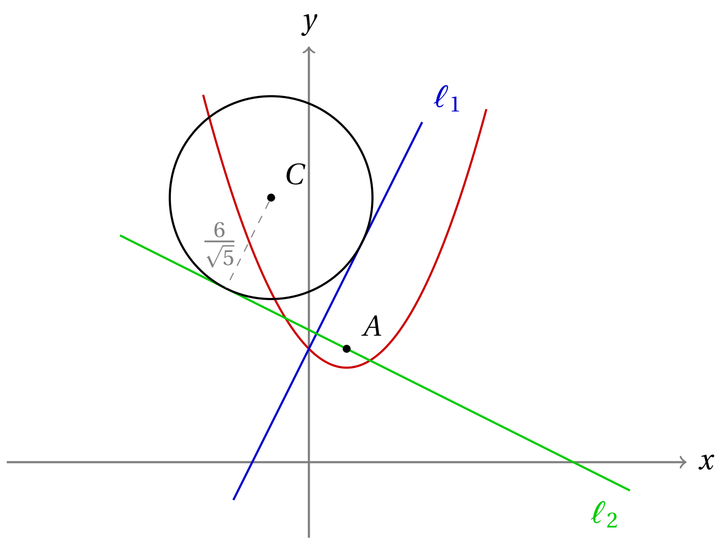


FIGURE 1: Illustration of the situation in question 3(d)

Squaring everything and multiplying throughout by 5, we get

$$(2a - b + 3)^2 = (a + 2b - 7)^2 = 36.$$

Here we have three equations in a and b; solving any two of them simultaneously gives the solution $a = -1$ and $b = 7$.

Thus the circle is $\boldsymbol{(x+1)^2 + (y-7)^2 = \frac{36}{5}}$.

4. (a) (i) We expand $R\cos(x - \alpha)$ using the compound angle identity for cosine, so that we may compare coefficients.

$$\begin{aligned} 3\cos x + \sqrt{3}\sin x &= R\cos(x - \alpha) \\ &= R\cos\alpha\cos x + R\sin\alpha\sin x, \end{aligned}$$

so we want that

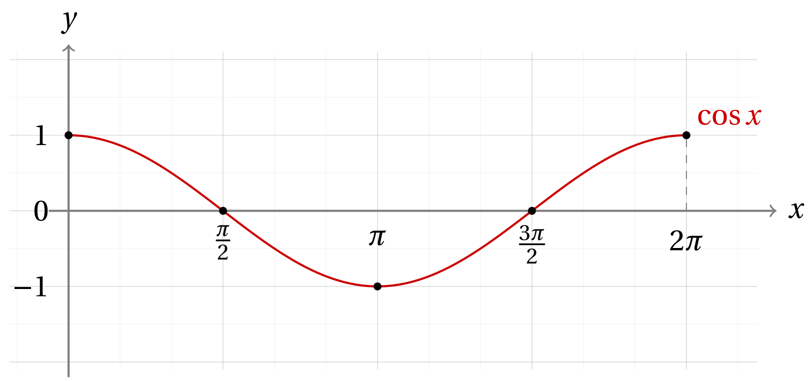
$$\begin{cases} R\cos\alpha = 3 & \text{①} \\ R\sin\alpha = \sqrt{3}. & \text{②} \end{cases}$$

If we do ② ÷ ①, we get $\tan\alpha = \frac{\sqrt{3}}{3}$, and we may take α to be the principal value $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$.

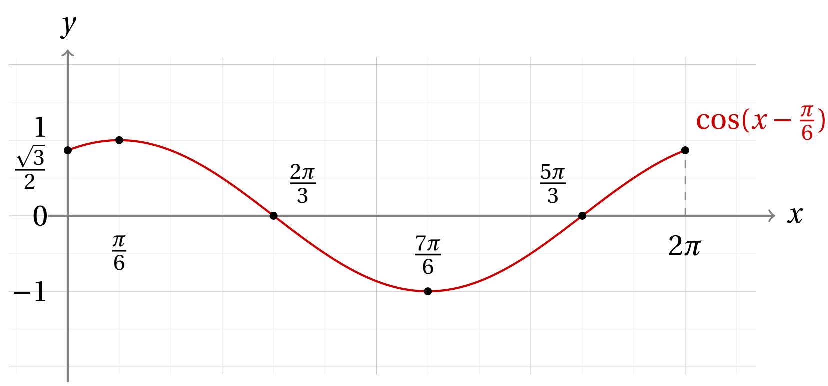
To find R, we take advantage of the Pythagorean identity, noting that ①^2 + ②^2 gives $R^2 = 12$, i.e., $R = 2\sqrt{3}$.

Thus $\boldsymbol{f(x) = 2\sqrt{3}\cos\left(x - \frac{\pi}{6}\right)}$.

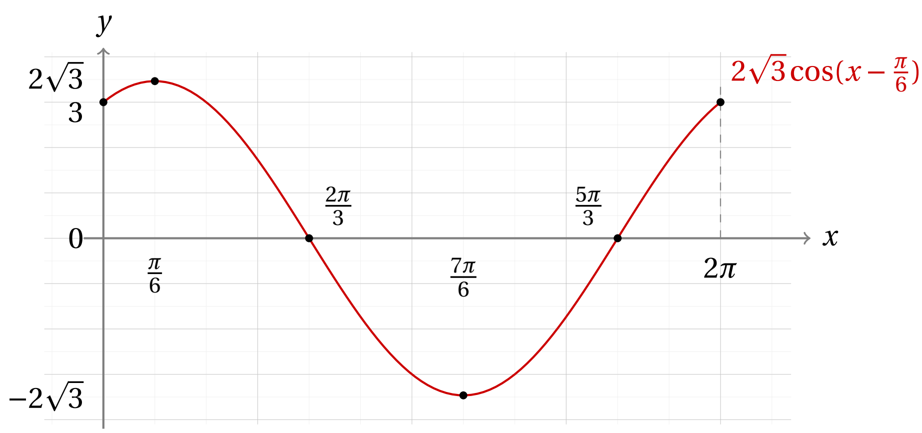
(ii) Now to sketch the graph of $y = 2\sqrt{3}\cos(x - \frac{\pi}{6})$, we start by drawing the regular cosine curve in the range $0 \leqslant x \leqslant 2\pi$:



Next, we apply the graphical transformation $f(x) \mapsto f(x - \frac{\pi}{6})$, which has the effect of translating the graph to the RIGHT by $\frac{\pi}{6}$ units. If we keep track of the x-intercepts, $\frac{\pi}{2}$ becomes $\frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$, and $\frac{3\pi}{2}$ becomes $\frac{5\pi}{3}$. Similarly, the peak at 0 moves to $\frac{\pi}{6}$, and the trough at π moves to $\frac{7\pi}{6}$. The y-intercept is now $\cos(0 - \frac{\pi}{6}) = \frac{\sqrt{3}}{2}$:



Finally, doing the transformation $f(x) \mapsto 2\sqrt{3}f(x)$ to this graph, we obtain the desired graph. This has the effect of a vertical stretch by a factor of $2\sqrt{3}$, so numbers on the y-axis are scaled appropriately. Thus the final sketch is:



(iii) To find this minimum, one simply needs to note that $\frac{1}{\textit{something}}$ is smallest (in size) whenever the *something* is largest (in size). Now in our case, the *something* is $f(x)+2$, and clearly this is largest whenever $f(x)$ is largest. To determine when $f(x)$ is largest, we simply need to look at the graph from part (ii): within the range $0 \leqslant x \leqslant \frac{\pi}{2}$, $f(x)$ reaches its maximum of $2\sqrt{3}$ at $x = \pi/6$. Thus the minimum of $1/(f(x)+2)$ is $1/(2\sqrt{3}+2) = \mathbf{\frac{1}{4}(\sqrt{3}-1)}$, and this occurs at $x = \boldsymbol{\frac{\pi}{6}}$.

(b) (i) There are many possible proofs, here is an easy one:

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta+\theta) \\
&= \cos(2\theta)\cos\theta - \sin 2\theta \sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\
&= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\
&= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta \\
&= 4\cos^3\theta - 3\cos\theta,
\end{aligned}$$

as required. □

(ii) If we put $x = \cos\theta$, the equation becomes

$$8\cos^3\theta - 6\cos\theta = 1 \implies 2\cos 3\theta = 1 \implies \cos 3\theta = \tfrac{1}{2}.$$

Solving in the usual way, we get the solutions

$$\theta = \pm 20^\circ, \pm 100^\circ, \pm 140^\circ$$

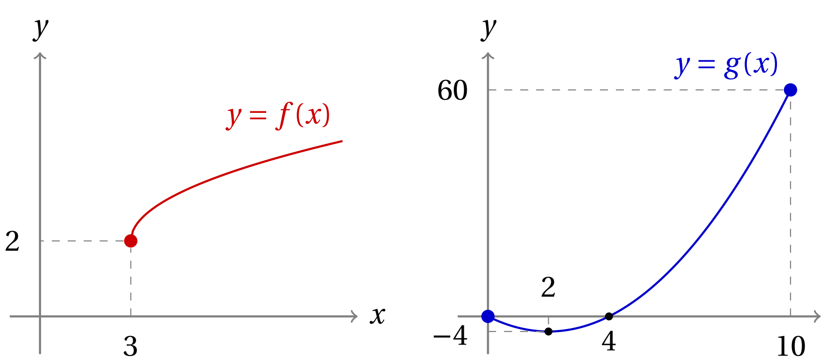
for θ in the range $-180^\circ < \theta \leqslant 180^\circ$. It suffices to solve over this range of values since $\cos\theta$ repeats itself every 360° (remember we put $x = \cos\theta$, so we care about the distinct values the output $\cos\theta$ takes on, not the input θ). Moreover, some of the solutions above are redundant: since cosine is even, we can disregard the negative values of θ since they will give us the same values of x as the positive ones.

Therefore the solutions are $\boldsymbol{x = \cos(20^\circ), \cos(100^\circ), \cos(140^\circ)}$.

Since we found three solutions, and the equation we have is a cubic, it cannot have any more, thus these are all its solutions.†

5. (a) Before solving the problems, it's good to sketch both $y = f(x)$ and $y = g(x)$ for reference. (This is the case with most exam questions on functions; especially when you spot that they are affinely transformed versions of the usual elementary functions you can sketch from memory.)

It's also good to determine the domain and range. For g, notice the minimum is $g(2) = -4$, and the maximum is $g(10) = 60$.



Domain:	$[3, \infty)$	Domain:	$[0, 10]$
Range:	$[2, \infty)$	Range:	$[-4, 60]$

†By letting $x = \cos\theta$, we are supposing that x can be expressed in this way in the first place; i.e., we are restricting x to be in the range of the cosine function, i.e., assuming that $x \in [-1, 1]$. If the equation happened to have any solutions larger than 1 (or smaller than -1), we would not have caught them this way.

(i) ***f* is injective**, since graphically, it is a translated version of $\sqrt{x}$, which is an injection (by the horizontal line test).

Alternatively, we see that

$$\begin{aligned} f(x) = f(y) &\implies 2 + \sqrt{x-3} = 2 + \sqrt{y-3} \\ &\implies \sqrt{x-3} = \sqrt{y-3} \\ &\implies x - 3 = y - 3 \\ &\implies x = y, \end{aligned}$$

i.e., f satisfies the defining property "$f(x) = f(y) \Rightarrow x = y$" of injectivity.

***g* is not injective**, from the sketch we see it fails the horizontal line test. Indeed, $g(0) = g(4) = 0$, i.e., the function g sends the distinct inputs $x = 0$ and $x = 4$ to the same output, $y = 0$, contradicting the definition of injection.

(ii) Neither are surjective, since neither have range $\mathbb{R}$ (notice the question indicated both have codomain $\mathbb{R}$).

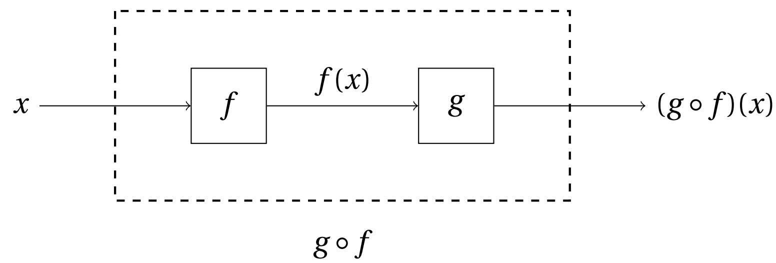
(iii) If we discard everything to the left of $x = 2$ in the graph of $y = g(x)$, we see that what we are left with is injective, so we take $\boldsymbol{k = 2}$.

Taking anything smaller than $k = 2$ will give a function which fails the horizontal line test, so it is the smallest k possible.

(b) For the expression, we simply apply g to $f(x)$. If we notice that $g(x) = (x-2)^2 - 4$ (by completing the square), the algebra ends up being quite simple:

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(2 + \sqrt{x-3}) \\ &= (2 + \sqrt{x-3} - 2)^2 - 4 \\ &= (x-3) - 4 \\ \therefore \boldsymbol{(g \circ f)(x)} &\boldsymbol{= x - 7}\,. \end{aligned}$$

Now to find its domain and range *as a composite function*, we need to think of $g \circ f$ as a "machine" made up of f and g:

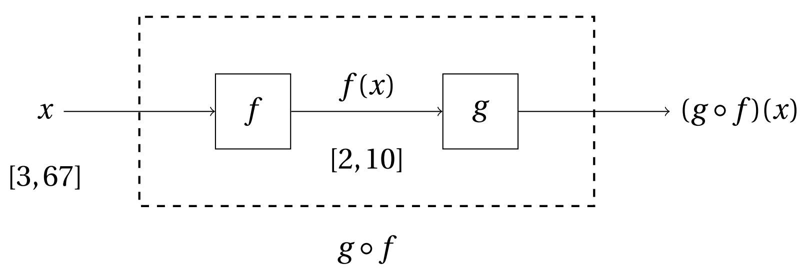


Thus, we need to make sure that the two components work well together: we need to take into account what g can take in, and what f possibly outputs, so that the values travelling along the "conveyor belt" in the middle live in $\text{ran}(f) \cap \text{dom}(g) = [2,\infty) \cap [0,10] = [2,10]$.

For f to only output things in the range $[2,10]$, from its graph, we see that we need to make sure it only accepts inputs below a certain point: namely, the point where it first outputs 10 (since f is always increasing).

$$f(x) = 10 \implies 2 + \sqrt{x-3} = 10 \implies \sqrt{x-3} = 8 \implies x = 3 + 64 = 67.$$

Thus, $f(67) = 10$, and from the graph of f, we see any input below 67 outputs something smaller than 10. Thus, if we let $x \in [3,67]$, f outputs things in $[2,10]$, as required:



Now for the range, we need to make a similar consideration: g will be receiving inputs in $[2,10]$, not its full domain (which is $[0,10]$), thus its range will be restricted to the outputs corresponding to inputs from 2 onwards. From the graph, in this case, we see that this will actually still be the same as the range of g, namely, $[-4,60]$.

Thus, $\mathbf{dom}(\boldsymbol{g \circ f}) = \mathbf{[3,67]}$ **and** $\mathbf{ran}(\boldsymbol{g \circ f}) = \mathbf{[-4,60]}$.