Mock Assessment Test

LUKE'S MATHS LESSONS*
Ħal Tarxien, Malta

Advanced Level PAPER I 23rd August, 2022

Instructions

The goal of this test is to prepare you for your MATSEC advanced level pure mathematics exam. The topics assessed here are those pertaining to the paper 1 syllabus.

Read the following instructions carefully.

- This test consists of **10 questions** and carries **100 marks**.
- You have **3 hours** to complete this test.
- Attempt **all** questions.

*https://maths.mt

1. Solve the differential equation

$$3\sqrt{y\sin x}\,dx + \sec x\,dy = 0$$

given that when $x = \frac{\pi}{2}$, $y = 0$. Give your answer in the form $y = f(x)$.

[10 marks]

2. Let $y = \cos^2(\log x)$.

   (a) Show that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y = 2.$$

   [Hint: use the identity $\cos^2 A = \frac{1}{2}(1+\cos 2A)$.]

   (b) Find $\int \frac{y}{x}\,dx$. **[6, 4 marks]**

3. (a) Find the equation of the line ℓ_1, joining the points having position vectors $3\mathbf{i} - \mathbf{j}$ and $5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, respectively.

   (b) The line ℓ_2 has equation $\mathbf{r} = \alpha\mathbf{i} - 2\mathbf{k} + \mu(\mathbf{i} + \beta\mathbf{j} - \mathbf{k})$. Find α and β, given that ℓ_1 and ℓ_2 are perpendicular and intersect each other.

[4, 6 marks]

4. (a) The line $\ell : ax + by + 10 = 0$ is tangent to the circle $\mathscr{C}$ with equation $(x+3)^2 + (y-4)^2 = 4$. Determine the constants a and b, given that they are integers, and that ℓ intersects the x-axis at $-\frac{5}{2}$.

   (b) Find the rotation matrix $\mathbf{R}$ associated with a (anticlockwise) rotation by an angle of $\sin^{-1}\frac{4}{5}$.

   (c) Find the image of $\mathscr{C}$ under $\mathbf{R}$, and verify that ℓ is still a tangent to it.

[4, 3, 3 marks]

5. Let $f(x) = x^2(1+x^4)\left(x^2 - \dfrac{1}{4x^2}\right)^{15}$.

   (a) Find the coefficient of x^{24} in the expansion of $f(x)$.

   (b) Solve the inequality $f(x) \geqslant 0$. **[5, 5 marks]**

6. (a) Express $f(\theta) = \sqrt{3}\cos 2\theta - \sin 2\theta$ in the form $R\cos(2\theta + \alpha)$, where $R > 0$ and $\alpha \in [0, \frac{\pi}{2}]$. Hence or otherwise, deduce the coordinates of the maximum turning point on the curve $y = f(\theta)$, for $0 \leqslant \theta \leqslant \pi$.

(b) Express

$$w = \frac{7-i}{1+2i}$$

in the form $a + bi$. Hence, given that w is a root of

$$f(z) = z^4 + 7z^2 + 18z + 10,$$

factorise $f(z)$ as much as possible using only real coefficients.

[5, 5 marks]

7. Consider the functions

$$f(x) = 1 + e^{x-1} \qquad \text{and} \qquad g(x) = 1 - \log(x+1),$$

where log denotes the natural logarithm.

(a) Sketch the graphs of $y = f(x)$ and $y = g(x)$ on separate diagrams.

(b) Find an expression for $(f \circ g)(x)$, and state the domain and range of the composition $f \circ g$.

(c) Show that when x is small, $(f \circ g)(x) \approx 2 - x + x^2$.
[Hint: use a series expansion.]

[3, 4, 3 marks]

8. (a) In how many ways can eight people be seated in a row of thirty chairs, such that no two people sit next to each other?

(b) Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$. Show that if a matrix $\mathbf{X}$ commutes with the matrix $\mathbf{A}$ (i.e., satisfies $\mathbf{AX} = \mathbf{XA}$) then it must be of the form $\begin{pmatrix} a & 3b \\ 2b & a \end{pmatrix}$ for appropriate constants a and b.

[5, 5 marks]

9. (a) Using integration by parts, or otherwise, find

$$\int (\theta + 1) \sec^2 \theta \, d\theta.$$

(b) Using an appropriate substitution, find

$$\int_2^4 \frac{\log(x^2)}{x (\log x)^2} \, dx,$$

where log denotes the natural logarithm.

[4, 6 marks]

10. (a) Find the two geometric progressions with first term 8 and sum of first three terms 14. Find the sum of the first n terms in both cases. State which of the two converges and find its sum to infinity.

(b) Solve the equation

$$\log_b(2x + k) = 2 \log_b(x) + k,$$

for x, assuming that $kb^k + 1 \geqslant 0$.

[5, 5 marks]

Answers

1. Separating the variables, the equation is

$$\frac{dy}{\sqrt{y}} = -3\sqrt{\sin x}\cos x\, dx,$$

and integrating both sides gives the general solution

$$2\sqrt{y} = -2\sin^{3/2} x + c.$$
$$\implies y(x) = (c - \sin^{3/2} x)^2.$$

Since $y = 0$ when $x = \frac{\pi}{2}$, we see that

$$0 = (c-1)^2 \implies c = 1,$$

thus the particular solution is

$$\boldsymbol{y(x) = (1 - \sin^{3/2} x)^2}.$$

2. (a) If $y = \cos^2(\log x)$, then

$$\begin{aligned} \frac{dy}{dx} &= 2\cos(\log x)\cdot - \sin(\log x)\cdot \tfrac{1}{x} \\ &= -\frac{\sin(2\log x)}{x}, \\ \frac{d^2y}{dx^2} &= -\frac{\cos(2\log x)\cdot \frac{2}{x}\cdot x - 1\cdot \sin(2\log x)}{x^2} \\ &= -\frac{2\cos(2\log x) - \sin(2\log x)}{x^2}. \end{aligned}$$

Thus

$$\begin{aligned} \text{LHS} &= x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y \\ &= -(2\cos(2\log x) - \sin(2\log x)) - \sin(2\log x) + 2(1 + \cos(2\log x)) \\ &= 2 = \text{RHS} \end{aligned}$$

□

(b) The integral is

$$\begin{aligned}\int \frac{\cos^2(\log x)}{x}\,dx &= \int \cos^2(\log x)\,d(\log x)\\ &= \frac{1}{2}\int (1+\cos(2\log x))\,d(\log x)\\ &= \boldsymbol{\tfrac{1}{4}(2\log x + \sin(2\log x))}.\end{aligned}$$

3. (a) Let the points be A and B, so that $\vec{OA} = 3\mathbf{i} - \mathbf{j}$ and $\vec{OB} = 5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$. Then $\vec{AB} = \vec{OB} - \vec{OA} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so we can take ℓ_1 to be

$$\begin{aligned} &\mathbf{r} = \vec{OA} + \lambda(\tfrac{1}{2}\vec{AB})\\ \therefore\quad &\boldsymbol{\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})}.\end{aligned}$$

(b) If ℓ_1 and ℓ_2 are perpendicular, it means their direction vectors are, i.e., $(\mathbf{i}+2\mathbf{j}-\mathbf{k})\cdot(\mathbf{i}+\beta\mathbf{j}-\mathbf{k}) = 0$, i.e., $1 + 2\beta + 1 = 0$, i.e., $\boldsymbol{\beta = -1}$.

Moreover, since ℓ_1 and ℓ_2 intersect, we have that for some λ and μ, they result in the same vector, i.e.,

$$\begin{aligned} &3\mathbf{i} - \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \alpha\mathbf{i} - 2\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} - \mathbf{k})\\ \implies\quad &(3+\lambda)\mathbf{i} + (-1+2\lambda)\mathbf{j} - \lambda\mathbf{k} = (\alpha+\mu)\mathbf{i} - \mu\mathbf{j} + (-2-\mu)\mathbf{k},\end{aligned}$$

comparing coordinates, we have that

$$\begin{cases} 3+\lambda = \alpha+\mu & ① \\ 1+2\lambda = -\mu & ② \\ \lambda = -2-\mu & ③ \end{cases}$$

Solving ② and ③ simultaneously, we get that $\lambda = -\mu = 1$, and from ②, it follows that $\alpha = 5$.

Thus $\boldsymbol{\alpha = 5, \beta = -1}$.

4. (a) Since ℓ intersects the x-axis at $-\frac{5}{2}$, it passes through $(-\frac{5}{2}, 0)$, i.e., $a(-\frac{5}{2}) + b(0) + 10 = 0$, i.e., $a = 4$.

Now being tangent to $\mathscr{C}$, the distance of ℓ from the centre $(-3,4)$ of the circle must be the radius, 2. In other words, we have

$$\begin{aligned} \frac{|4(-3)+b(-4)+10|}{\sqrt{4^2+b^2}} &= 2 \\ \Longrightarrow \qquad (4b+2)^2 &= 4(b^2+16) \\ \Longrightarrow \qquad (b-3)(3b-5) &= 0 \\ \Longrightarrow \qquad b &= 3, \end{aligned}$$

since the constants should be integers. Thus $\boldsymbol{a = 4, b = 3}$.

(b) The rotation matrix about any general angle θ is

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

With $\theta = \sin^{-1}\frac{4}{5}$, we can compute the entries as $\sin\theta = \frac{4}{5}$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}$, thus

$$\mathbf{R} = \mathbf{\frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}}.$$

(c) Since $\mathbf{R}$ is a rotation, $\mathscr{C}$ will remain a circle after being transformed; we just need to see where the centre goes:

$$\mathbf{R}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}.$$

Thus the image of $\mathscr{C}$ is a circle of the same radius, centred at $(-5,0)$, i.e., $(x+5)^2 + y^2 = 4$.

Now to check that it is still tangent to the line, we can verify that the line is a distance of 2 away from its centre $(-5,0)$. Indeed,

$$d(\ell,(-5,0)) = \frac{|4(-5)+3(0)+10|}{\sqrt{4}^2+3^2} = \frac{|-10|}{5} = 2,$$

as required. □

5. (a) Rearranging $f(x)$ a bit, we have

$$\begin{aligned} f(x) &= x^2(1+x^4)\left(x^2 - \frac{1}{4x^2}\right)^{15} \\ &= x^2(1+x^4)(4^{-1}x^{-2}(4x^4-1))^{15} \\ &= 4^{-15}x^{-28}(1+x^4)(4x^4-1)^{15} \\ &= 4^{-15}(x^{-28}+x^{-24})(4x^4-1)^{15}. \end{aligned}$$

Now the only contribution to the coefficient of x^{24} must come from the coefficients of x^{52} and x^{48} in the expansion of $(4x^4-1)^{15}$, since then multiplying these by x^{-28} and x^{-24} respectively will give x^{24}.

In other words, the part of the expansion we care about is

$$\begin{aligned} &4^{-15}(x^{-28}+x^{-24})\left(\cdots + \binom{15}{12}(4x^4)^{12}(-1)^3 + \binom{15}{13}(4x^4)^{13}(-1)^2 + \cdots\right) \\ &= 4^{-15}(x^{-28}+x^{-24})(-455\cdot 4^{12}x^{48} + 105\cdot 4^{13}x^{52} + \cdots), \end{aligned}$$

and more specifically, expanding the two brackets, we see that the two terms having terms in x^{24} are

$$4^{-15}(-455\cdot 4^{12}x^{24} + 105\cdot 4^{13}x^{24}) = -\frac{455}{4^3}x^{24} + \frac{105}{4^2}x^{24} = -\frac{35}{64}x^{24},$$

thus the coefficient is $-\mathbf{\frac{35}{64}}$.

(b) Notice that the expressions x^2 and $(1+x^4)$ are always non-negative, so they have no bearing on whether $f(x)$ is $\geqslant 0$ or not. Moreover, the sign of $\left(x^2 - \frac{1}{4x^2}\right)^{15}$ is the same as the sign of the inner term, $x^2 - \frac{1}{4x^2}$. Thus the inequality $f(x) \geqslant 0$ amounts to just asking when

$$\begin{aligned} & x^2 - \tfrac{1}{4x^2} \geqslant 0 \\ \Longrightarrow\quad & 4x^4 - 1 \geqslant 0 \\ \Longrightarrow\quad & (2x^2+1)(2x^2-1) \geqslant 0 \qquad (\div 2x^2+1) \\ \Longrightarrow\quad & 2x^2 - 1 \geqslant 0, \end{aligned}$$

and drawing a quick sketch of the parabola $2x^2-1$, we see that it is non-negative for $\boldsymbol{x \leqslant -\frac{\sqrt{2}}{2}}$ **or** $\boldsymbol{x \geqslant \frac{\sqrt{2}}{2}}$.

6. (a) We expand the expression $R\cos(2\theta+\alpha)$ using the compound angle identity for cosine, so that we may compare coefficients.

$$\begin{aligned}\sqrt{3}\cos 2\theta - \sin 2\theta &= R\cos(2\theta+\alpha) \\ &= R\cos\alpha\cos 2\theta - R\sin\alpha\sin 2\theta,\end{aligned}$$

so we want that

$$\begin{cases} R\cos\alpha = \sqrt{3} & \text{①} \\ R\sin\alpha = 1. & \text{②} \end{cases}$$

If we do ② ÷ ①, we get $\tan\alpha = \frac{\sqrt{3}}{3}$, and we may take α to be the principal value $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$.

To find R, we take advantage of the Pythagorean identity, noting that ①^2 + ②^2 gives $R^2 = 4$, i.e., $R = 2$.

Thus $\boldsymbol{f(\theta) = 2\cos\left(2\theta + \frac{\pi}{6}\right)}$.

Now, the maximum turning point occurs when $2\cos(2\theta+\frac{\pi}{6}) = 2$, i.e., when $\cos(2\theta+\frac{\pi}{6}) = 1$. We can use the general solution, or just recall from the graph that $\cos\theta$ is 1 at even multiples of π, we see that

$$\begin{aligned} & 2\theta + \tfrac{\pi}{6} = 2n\pi \\ \implies \quad & \theta = n\pi - \tfrac{\pi}{12}, \end{aligned}$$

and it's clear the only value of θ in the desired range is obtained when we put $n = 1$, which is $\theta = \frac{11}{12}\pi$. Thus the maximum turning point occurs at $\boldsymbol{\left(\frac{11}{12}\pi, 2\right)}$.

(b) Multiplying the top and bottom by the conjugate $(1+2i)^* = 1-2i$, we get that

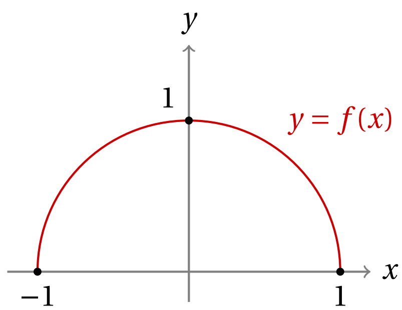
$$w = \frac{(7-i)(1-2i)}{1^2+2^2} = 1-3i.$$

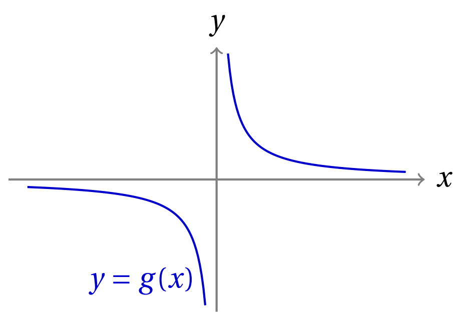
Now, since f has real coefficients, any complex roots it has must occur in conjugate pairs. Thus $w^* = 1+3i$ must also be a root. Since $w + w^* = 2$ and $ww^* = 1^2+3^2 = 10$, we see that $z^2-2z+10$ is the quadratic with w, w^* as its roots. Thus $(z^2-2z+10) \mid f$, and we can write

$$\boldsymbol{f(z) = (z^2-2z+10)(z+1)^2}$$

using long division.

7. Notice that f is the top-half of the circle $x^2 + y^2 = 1$.





Domain:	$[-1,1]$		Domain:	$\mathbb{R} \smallsetminus \{0\}$
Range:	$[0,1]$		Range:	$\mathbb{R} \smallsetminus \{0\}$

(a) We have

$$h_1(x) = (f \circ g)(x) = \sqrt{1 - \frac{1}{x^2}},$$

and

$$\begin{aligned} \operatorname{dom}(h_1) &= g^{-1}(\operatorname{ran}(g) \cap \operatorname{dom}(f)) \\ &= g^{-1}([-1,1] \smallsetminus \{0\}) \\ &= (-\infty,-1] \cup [1,\infty). \end{aligned}$$

From their graphs, we see that if we restrict h_1 to $[1,\infty)$, then h_1 is one-to-one.

(b) Similarly,

$$h_2(x) = (g \circ f)(x) = \frac{1}{\sqrt{1-x^2}},$$

and

$$\begin{aligned} \operatorname{dom}(h_2) &= f^{-1}(\operatorname{ran}(f) \cap \operatorname{dom}(g)) \\ &= f^{-1}((0,1]) \\ &= [-1,1] \smallsetminus \{0\}. \end{aligned}$$

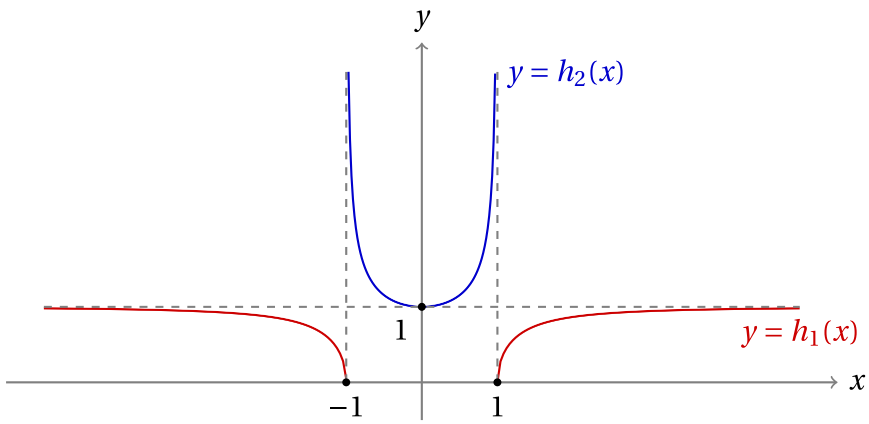
From the graphs, we see that h_2 is one-to-one on $(0,1]$.

(c) We have

$$(h_1 \circ h_2)(x) = \sqrt{1 - \frac{1}{\left(\frac{1}{\sqrt{1-x^2}}\right)^2}} = \sqrt{1-(1-x^2)} = \sqrt{x^2} = x,$$

so h_1 and h_2 are mutual inverses.

Sketch:



8. (a) There are 18 letters, with the following repetitions: 4 T's, 3 A's, 3 N's, 2 I's and 2 S's. Thus the number of permutations is

$$\frac{18!}{4!3!3!2!2!} = \mathbf{1\,852\,538\,688\,000}.$$

(b) Let **V** represent the place where all the vowels will go. Then, ignoring the vowels for now, the permutations we desire are all those of the word TRNSBSTNTTN**V**, which has

$$\frac{12!}{4!3!2!} = 1\,663\,200 \text{ permutations.}$$

Now, we replace the token letter **V** with all the vowels (this way, they remain together). The vowels from the original word are AUAIAIO, which among themselves, have

$$\frac{7!}{3!2!} = 420 \text{ permutations.}$$

Thus, overall, we have $1\,663\,200 \times 420 = \mathbf{698\,544\,000}$ permutations.

(c) The probability is

$$\frac{698\,544\,000}{1\,852\,538\,688\,000} = \mathbf{\frac{1}{2652}}.$$

9. (a) We have

$$\int \frac{\cos\theta}{\sin^2\theta + 3}\,d\theta = \int \frac{d(\sin\theta)}{\sin^2\theta+3} = \boldsymbol{\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sin\theta}{\sqrt{3}}\right) + c}.$$

(b) Integrating by parts,

$$\begin{aligned}\int_1^e \frac{\log x}{x^2}\,dx &= \int_1^e \log x\, d\left(-\tfrac{1}{x}\right)\\ &= \left[-\frac{\log x}{x}\right]_1^e + \int_1^e \frac{d(\log x)}{x}\\ &= -\frac{1}{e} + \int_1^e \frac{dx}{x^2}\\ &= -\frac{1}{e} + \left[-\frac{1}{x}\right]_1^e\\ &= \boldsymbol{1 - \frac{2}{e}}.\end{aligned}$$

10. (a) We have $a = 8$ and $a + ar + ar^2 = 14$, and plugging one into the other gives $8 + 8r + 8r^2 = 14$, i.e., $1 + r + r^2 = \frac{7}{4}$. This quadratic has roots $\frac{1}{2}$ and $-\frac{3}{2}$.

Thus the two geometric progressions are $a_n = 8\left(\frac{1}{2}\right)^{n-1} = 16\cdot 2^{-n}$ and $b_n = 8\left(-\frac{3}{2}\right)^{n-1} = \frac{16}{3}\left(-\frac{3}{2}\right)^n$.

The sum of the first n terms of a_n is

$$\sum_{k=1}^{n} a_k = 8\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 16(1-2^{-n})$$

and similarly for b_n, we have

$$\sum_{k=1}^{n} b_k = 8\frac{1-\left(-\frac{3}{2}\right)^n}{1--\frac{3}{2}} = \frac{16}{5}\left(1-\left(-\tfrac{3}{2}\right)^n\right).$$

Since only a_n satisfies $|r| < 1$, its sum is the convergent one. Indeed, we have

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} 16(1-2^{-n}) = \mathbf{16}.$$

(b) Using laws of logarithms, the equation is

$$\log_b(2x+k) = \log_b(x^2 b^k)$$

which is equivalent to the quadratic

$$b^k x^2 - 2x - k = 0,$$

which has solutions

$$\boldsymbol{x = \tfrac{1}{b^k}\left(1 \pm \sqrt{1+kb^k}\right)}.$$