Mock Assessment Test

MATHEMATICS TUTORIALS*
Ħal Tarxien, Malta

Advanced Level May 2021

Instructions

The goal of this test is to prepare you for your MATSEC advanced level pure mathematics exam. The topics covered are those examinable in paper 1.

Read the following instructions carefully.

- This test consists of **6 questions** and carries **60 marks**.
- You have **two hours** to complete this test.
- Attempt **all** questions.

*https://maths.mt

1. (a) Resolve the function $\dfrac{3x^2-2x+11}{(x^2+1)(x-4)}$ into partial fractions.

(b) Solve the differential equation

$$(x^2+1)(x-4)\frac{dy}{dx} = \sec^4 y\,(3x^2-2x+11),$$

given that $y=0$ when $x=5$.

[3, 7 marks]

2. (a) Find the equation of the circle $\mathscr{C}$, having centre lying on the x-axis, whose tangent at the point $(2,1)$ is given by the equation $\ell_1 : 2y = x$.

(b) Verify that $\ell_2 : 2y = 5 - x$ is also a tangent to the circle.

(c) Find the point of intersection of ℓ_1 and ℓ_2, and verify (for this case) the classical geometry theorem which states that *tangents to a circle from the same point are equal in length.*

[5, 2, 3 marks]

3. (a) (i) Express $f(x) = \cos x + \sqrt{3}\sin x$ in the form

$$f(x) = R\cos(x-\alpha)$$

where $R > 0$ and $\alpha \in \left[0, \frac{\pi}{2}\right]$.

(ii) Sketch $y = f(x)$ in the range $[0, 2\pi]$, clearly indicating the curve's amplitude, and the points where it intersects the coordinate axes.

(iii) Find the minimum value of $1/(f(x)+1)$ for $0 \leqslant x \leqslant \pi$, and state the value of x at which this minimum occurs.

(b) Determine the five solutions to the equation

$$\sin\theta + \sin 3\theta + \sin 5\theta = 0$$

for θ in the range $0 < \theta < 360°$.

[6, 4 marks]

4. (a) Find dy/dx in terms of x and y for each of the following.

   (i) $x = e^t \cos t$, $y = e^t \sin t$, where $t \in \mathbb{R}$ is a parameter.

   (ii) $4x^2 - 3xy + 5y^3 + 2\cos y \sin x = 7$.

   (b) Find the tangent to the curve $y = \exp(\sin^2 x - \cos x)$ at the point where $x = \frac{\pi}{2}$. [Note: $\exp x \equiv e^x$]

   (c) If x and y are nonnegative real numbers such that $x + y = 15$, what is the maximum possible value of xy^2?

[4, 3, 3 marks]

5. (a) (i) Express $p(x) = 3 - 2x - x^2$ in the form $a - (x+b)^2$.

   (ii) Use the substitution $x + 1 = 2\sin u$ to show that

$$\int_0^1 \sqrt{3 - 2x - x^2}\, dx = \frac{4\pi - 3\sqrt{3}}{6}.$$

   (b) Find

$$\int e^{\alpha\theta} \cos \beta\theta \, d\theta.$$

[5, 5 marks]

6. The real-valued function f is given by

$$f(x) = \begin{cases} 1 - x & \text{for } x \leqslant -2 \\ a - x - x^2 & \text{for } -2 \leqslant x \leqslant 2 \\ \ln(x+b) & \text{for } x \geqslant 2 \end{cases}$$

   (a) Determine the constants a and b so that f is continuous (i.e., the endpoints of each piece of the function meet at the same point).

   (b) Sketch f in the range $-5 < x < 5$, clearly indicating the intercepts with the coordinate axes, and state its domain and range.

   (c) The real-valued function g is given by $g(x) = 3 - |x|$ over the domain $-3 < x < 1$. Sketch the function $g(x)$ and state its range. Also, show that there is no point where $f(x) = g(x)$.

[3, 3, 4 marks]

Hints and Solutions

1. (a) We have the partial fraction expansion

$$\frac{3x^2-2x+11}{(x^2+1)(x-4)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-4}.$$

For C, we multiply throughout by $(x-4)$ to get

$$\frac{3x^2-2x+11}{x^2+1} = \frac{(Ax+B)(x-4)}{x^2+1} + C,$$

and putting $x=4$, we have

$$\frac{3(4)^2-2(4)+11}{(4)^2+1} = 0 + C,$$

hence $C = 3$.

Now, for A and B:

$$\begin{aligned}
\frac{3x^2-2x+11}{(x^2+1)(x-4)} &= \frac{Ax+B}{x^2+1} + \frac{3}{x-4} \\
&= \frac{(Ax+B)(x-4)+3(x^2+1)}{(x^2+1)(x-4)} \\
&= \frac{(3+A)x^2+(B-4A)x+3-4B}{(x^2+1)(x-4)} \\
\implies 3x^2-2x+11 &= (3+A)x^2+(B-4A)x+3-4B
\end{aligned}$$

Since we have two identical polynomials in x, their coefficients must be equal. Comparing the coefficients of x^2, we get $3 = 3+A$, so $A = 0$. Now comparing the coefficients of x^0 (i.e., the constant terms), we have $11 = 3-4B \implies 8 = -4B$, so $B = -2$.

$$\therefore \boxed{\frac{3x^2-2x+11}{(x^2+1)(x-4)} = \frac{3}{x-4} - \frac{2}{x^2+1}}.$$

(b) $(x^2+1)(x-4)\dfrac{dy}{dx} = \sec^4 y\,(3x^2-2x+11)$

$$\implies \frac{dy}{\sec^4 y} = \frac{3x^2-2x+11}{(x^2+1)(x-4)}\,dx$$

$$\implies \int \cos^4 y\,dy = \int \left(\frac{3}{x-4} - \frac{2}{x^2+1}\right) dx \qquad \text{(by part (a) above)}$$

$$\implies \int \cos^4 y\,dy = 3\int \frac{1}{x-4}\,dx - 2\int \frac{1}{x^2+1}\,dx$$

$$\implies \int (\cos^2 y)^2\,dy = 3\log(x-4) - 2\tan^{-1} x \qquad \circledast$$

Now, continuing to evaluate the LHS:

$$\begin{aligned}\int (\cos^2 y)^2\,dy &= \int \left(\frac{1}{2}(\cos 2y+1)\right)^2 dy \\ &= \frac{1}{4}\int \left(\cos^2 2y + 2\cos 2y + 1\right) dy \\ &= \frac{1}{4}\int \cos^2 2y\,dy + \frac{1}{2}\int \cos 2y\,dy + \frac{1}{4}\int dy \\ &= \frac{1}{4}\int \frac{1}{2}(\cos 4y+1)\,dy + \frac{\sin 2y}{4} + \frac{y}{4} \\ &= \frac{\sin 4y}{32} + \frac{y}{8} + \frac{\sin 2y}{4} + \frac{y}{4} + c\end{aligned}$$

Substituting back in $\circledast$, we have

$$\begin{aligned}&\frac{\sin 4y}{32} + \frac{y}{8} + \frac{\sin 2y}{4} + \frac{y}{4} + c = 3\log(x-4) - 2\tan^{-1} x \\ \implies& \sin 4y + 8\sin 2y + 12y + c = 96\log(x-4) - 64\tan^{-1} x\end{aligned}$$

as the *general solution.* Now we can obtain the particular solution by determining the constant c to match the initial condition $y(5) = 0$:

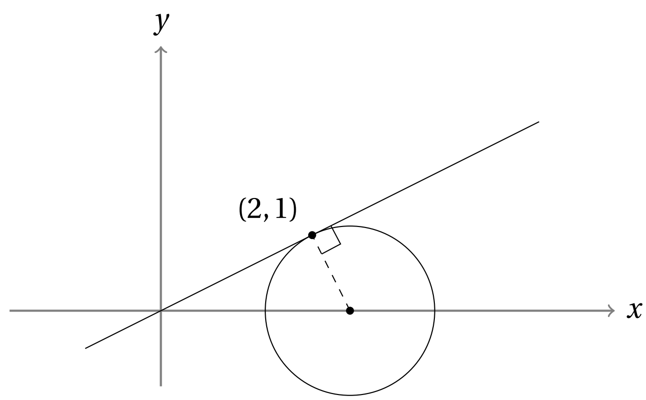
$$\left.\begin{matrix} x=5 \\ y=0 \end{matrix}\right\} \implies \sin 0 + 8\sin 0 + 12(0) + c = 96\log 1 - 64\tan^{-1} 5$$

$$\implies c = -64\tan^{-1} 5.$$

Therefore the *particular solution* is

$$\boxed{\sin 4y + 8\sin 2y + 12y = 96\log(x-4) - 64(\tan^{-1}x + \tan^{-1}5)}.$$

2. We have the following scenario:



(a) We need to find the equation of a circle—this involves finding its centre (a, b) and its radius r, then substituting these into the equation $(x-a)^2 + (y-b)^2 = r^2$.

Let us start with the centre. Since the circle is centred at the origin, then the normal to the circle at $(2, 1)$ (i.e., the radius line in the diagram) intersects the x-axis at the centre. Now the given tangent has gradient

$$m = \frac{1-0}{2-0} = \frac{1}{2},$$

so the normal at the same point has gradient $m' = -\frac{1}{m} = -2$.

Hence the equation of the normal is $y - 1 = -2(x-2) \implies y = 5 - 2x$. Now that we have this equation, substituting $y = 0$ will give us the x-intercept. $y = 0 \implies 0 = 5 - 2x \implies 2x = 5 \implies x = \frac{5}{2}$. Thus the coordinates of the centre are $(\frac{5}{2}, 0)$.

Now the radius is easy to find: we know the centre, and we know a point on the circumference. Thus r is simply the distance between these two points:

$$r = \sqrt{\left(2 - \frac{5}{2}\right)^2 + (1-0)^2} = \frac{\sqrt{5}}{2}.$$

Therefore the equation of the circle $\mathscr{C}$ is $\boxed{\left(x-\frac{5}{2}\right)^2+y^2=\frac{5}{4}}$.

(b) $\ell_2 : 2y = 5 - x$ is a tangent to the circle $\mathscr{C}$ if and only if they intersect at precisely one point. We can show that this is the case by solving their equations simultaneously. We have the system

$$\begin{cases} \left(x-\frac{5}{2}\right)^2+y^2=\frac{5}{4} & \text{①} \\ 2y = 5-x & \text{②} \end{cases}$$

From ②, we get that $y = (5-x)/2$. Substituting this in ① gives

$$\begin{aligned} &\left(x-\frac{5}{2}\right)^2+\left(\frac{5-x}{2}\right)^2=\frac{5}{4} \\ \implies &x^2-5x+\frac{25}{4}+\frac{x^2-10x+25}{4}=\frac{5}{4} \\ \implies &5x^2-30x+50=5 \\ \implies &x^2-6x+9=0 \\ \implies &(x-3)^2=0 \\ \implies &x=3 \quad \text{(twice)} \end{aligned}$$

By ②, we then get the corresponding y-coordinate $y = (5-3)/2 = 1$, so the circle and line only intersect at $(3,1)$, proving tangency. □

(c) We solve the system

$$\begin{cases} \ell_1 : 2y = x & \text{①} \\ \ell_2 : 2y = 5-x & \text{②} \end{cases}$$

to find the point of intersection. ① − ② gives $0 = 2x-5$, so $x = \frac{5}{2}$. Substituting into ①, we get $y = \frac{x}{2} = \frac{5}{4}$. Hence ℓ_1 and ℓ_2 intersect at $\boxed{\left(\frac{5}{2},\frac{5}{4}\right)}$.

Now to verify that "*tangents from the same point are equal in length*", we show that the distance from the intersection point $\left(\frac{5}{2},\frac{5}{4}\right)$ to the point of contact with the circle is equal for both tangents. Let d_1, d_2 represent these distances for ℓ_1 and ℓ_2 respectively.

Since ℓ_1 intersects the circle at $(2,1)$,

$$d_1 = \sqrt{\left(\frac{5}{2}-2\right)^2 + \left(\frac{5}{4}-1\right)^2} = \frac{\sqrt{5}}{4}.$$

Now recall from part (b) that ℓ_2 intersects the circle at $(3,1)$. Thus the distance

$$d_2 = \sqrt{\left(\frac{5}{2}-3\right)^2 + \left(\frac{5}{4}-1\right)^2} = \frac{\sqrt{5}}{4} = d_1,$$

as required. □

3. (a) (i) We expand $R\cos(x-\alpha)$ using the compound angle identity for cosine, so that we may compare coefficients.

$$\begin{aligned}\cos x + \sqrt{3}\sin x &= R\cos(x-\alpha)\\ &= R\cos\alpha\cos x + R\sin\alpha\sin x,\end{aligned}$$

so we want that

$$\begin{cases} R\cos\alpha = 1 & \textcircled{1}\\ R\sin\alpha = \sqrt{3}. & \textcircled{2}\end{cases}$$

If we do ② ÷ ①, we get $\tan\alpha = \sqrt{3}$, and we may take α to be the principal value $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$. To find R, we take advantage of the Pythagorean identity, noting that ①^2 + ②^2 gives $R^2 = 4$, i.e., $R = 2$.

Thus we have

$$\cos x + \sqrt{3}\sin x = \boxed{2\cos(x-\tfrac{\pi}{3})}.$$

(ii) Now to sketch the graph of $y = 2\cos(x-\frac{\pi}{3})$, we start by drawing the regular cosine curve in the range $0 \leqslant x \leqslant 2\pi$ (figure 1). Next, we apply the graphical transformation $f(x) \mapsto f(x-\frac{\pi}{3})$, which has the effect of translating the graph to the *right* by $\frac{\pi}{3}$ units. If we keep track of the x-intercepts, $\frac{\pi}{2}$ becomes $\frac{\pi}{2}+\frac{\pi}{3} = \frac{5\pi}{6}$, and $\frac{3\pi}{2}$ becomes $\frac{11\pi}{6}$. Similarly, the peak at 0 moves to $\frac{\pi}{3}$, and the trough

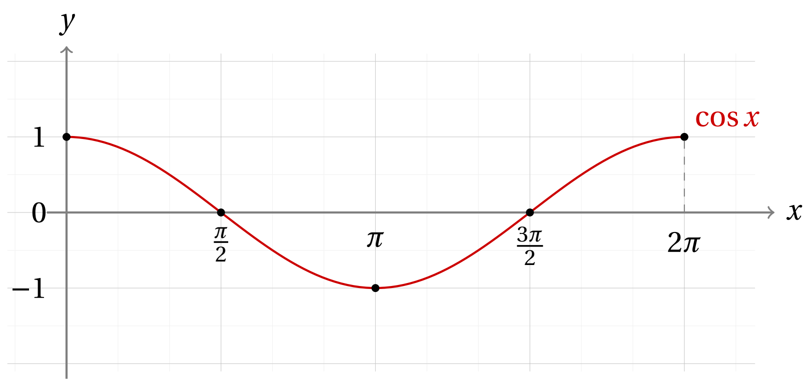


FIGURE 1: Sketch of $y = \cos x$

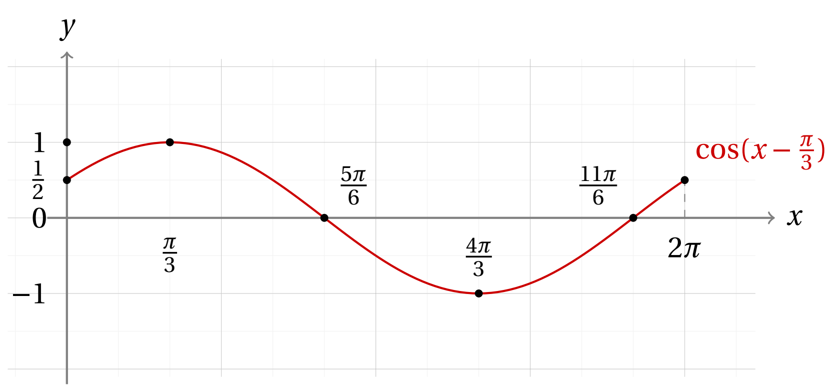


FIGURE 2: Sketch of $y = \cos(x + \frac{\pi}{3})$

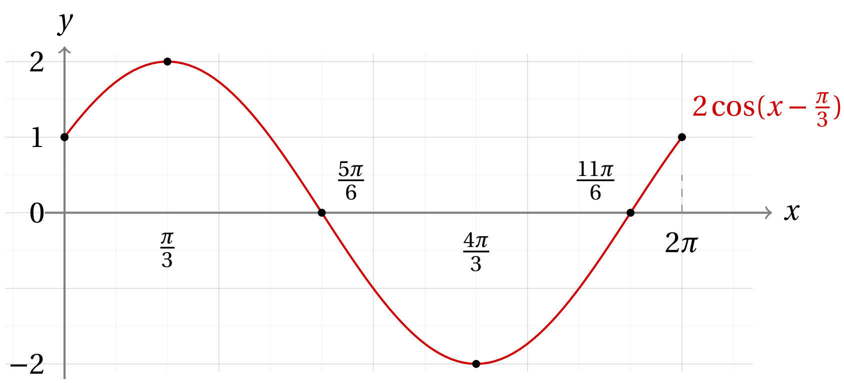


FIGURE 3: Sketch of $y = 2\cos(x + \frac{\pi}{3})$

at π moves to $\frac{4\pi}{3}$. The y-intercept is now $\cos(0 - \frac{\pi}{3}) = \frac{1}{2}$ (figure 2). Finally, doing the transformation $f(x) \mapsto 2f(x)$ to this graph, we obtain the desired graph. This has the effect of a vertical stretch by a factor of two, so numbers on the y-axis are doubled (figure 3).

(iii) To find this minimum, one simply needs to note that $\frac{1}{\textit{something}}$ is smallest (in size) whenever the *something* is largest (in size). Now in our case, the *something* is $f(x) + 1$, and clearly this is largest whenever $f(x)$ is largest. To determine when $f(x)$ is largest, we simply need to look at the graph from part (a): within the range $0 \leqslant x \leqslant \pi$, $f(x)$ reaches its maximum of 2 at $x = \pi/3$. Thus the minimum of $1/(1 + f(x))$ is $1/(1+2) = \frac{1}{3}$, and this occurs at $x = \frac{\pi}{3}$.

(b) Noting that the average of θ and 5θ is 3θ, one way we can proceed here is to use the *sum-to-product* trigonometric identities.

$$
\begin{aligned}
& \sin\theta + \sin 3\theta + \sin 5\theta = 0 \\
\Longrightarrow\ & \sin\theta + \sin 5\theta + \sin 3\theta = 0 \\
\Longrightarrow\ & 2\sin\left(\tfrac{\theta+5\theta}{2}\right)\cos\left(\tfrac{\theta-5\theta}{2}\right) + \sin 3\theta = 0 \\
\Longrightarrow\ & 2\sin 3\theta \cos 2\theta + \sin 3\theta = 0 \\
\Longrightarrow\ & \sin 3\theta(2\cos 2\theta + 1) = 0 \\
\Longrightarrow\ & \sin 3\theta = 0 \quad \text{or} \quad 2\cos 2\theta + 1 = 0
\end{aligned}
$$

Solving both equations separately, we first have have

$$\begin{aligned}
&\sin 3\theta = 0\\
\implies &(3\theta)_{\text{pv}} = \sin^{-1} 0 = 0\\
\implies &3\theta = 180^\circ n, \quad n \in \mathbb{Z}\\
\implies &\theta = 60^\circ n, \quad n \in \mathbb{Z}
\end{aligned}$$

Taking $n = 1, 2, 3, 4, 5$, the values of θ in range are 60°, 120°, 180°, 240°, and 300°. Next we solve the second equation

$$\begin{aligned}
&2\cos 2\theta + 1 = 0\\
\implies &\cos 2\theta = -\frac{1}{2}\\
\implies &(2\theta)_{\text{pv}} = \cos^{-1}\left(-\tfrac{1}{2}\right) = 120^\circ\\
\implies &2\theta = 360^\circ n \pm 120^\circ, \quad n \in \mathbb{Z}\\
\implies &\theta = 180^\circ n \pm 60^\circ, \quad n \in \mathbb{Z}
\end{aligned}$$

Taking $n = 0, 1, 2$, we obtain $\theta = 60^\circ, 120^\circ, 240^\circ, 300^\circ$ in range. Thus combining the results from both cases, we get the solutions

$$\boxed{\theta = 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ}.$$

4. (a) (i)

$$\begin{aligned}
&x = e^t \cos t & &y = e^t \sin t\\
\implies &\frac{dx}{dt} = e^t \cos t - e^t \sin t & \implies &\frac{dy}{dt} = e^t \sin t + e^t \cos t\\
&= x - y & &= y + x
\end{aligned}$$

By the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = (y + x)\dfrac{1}{x - y}$, so $\boxed{\dfrac{dy}{dx} = \dfrac{x + y}{x - y}}$.

(ii)

$$\begin{aligned}
&4x^2 - 3xy + 5y^3 + 2\cos y \sin x = 7\\
\implies &\frac{d}{dx}(4x^2 - 3xy + 5y^3 + 2\cos y \sin x) = \frac{d}{dx}(7)\\
\implies &8x - 3y - 3x\frac{dy}{dx} + 15y^2\frac{dy}{dx} + 2\cos y \cos x - 2\sin y \sin x\frac{dy}{dx} = 0\\
\implies &\frac{dy}{dx}(15y^2 - 3x - 2\sin y \sin x) = 3y - 8x - 2\cos y \cos x\\
\therefore &\boxed{\frac{dy}{dx} = \frac{3y - 8x - 2\cos y \cos x}{15y^2 - 3x - 2\sin y \sin x}}
\end{aligned}$$

(b) $y = \exp(\sin^2 x - \cos x)$

$$\implies \frac{dy}{dx} = (2\sin x\cos x + \sin x)\exp(\sin^2 x - \cos x)$$

$$\implies \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = \left(2\sin\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right)\exp\left(\sin^2\frac{\pi}{2} - \cos\frac{\pi}{2}\right) = \exp 1 = e$$

Now, at $x = \frac{\pi}{2}$, $y = \exp(\sin^2\frac{\pi}{2} - \cos\frac{\pi}{2}) = \exp 1 = e$. Thus the equation of the tangent is

$$\boxed{y - e = e\left(x - \frac{\pi}{2}\right)}$$

(c) $x + y = 15$, we want to find $\max(xy^2)$.

$$\text{Set } \frac{d}{dy}(xy^2) = 0 \qquad \text{(for critical values)}$$

$$\implies \frac{d}{dy}((15-y)y^2) = 0 \qquad (\text{since } x + y = 15)$$

$$\implies \frac{d}{dy}(15y^2 - y^3) = 0$$

$$\implies 30y - 3y^2 = 0$$

$$\implies y(10 - y) = 0$$

$$\implies y = 0 \qquad \text{or} \qquad y = 10$$

Thus the maximum occurs when $y = 10$, so $\max(xy^2) = (15-10)10^2 = \boxed{500}$.

5. (a) (i) By completing the square, we have

$$\begin{aligned} p(x) &= 3 - 2x - x^2 \\ &= -(x^2 + 2x - 3) \\ &= -[(x+1)^2 - 3 - 1] \\ &= -[(x+1)^2 - 4] \\ &= 4 - (x+1)^2 \end{aligned}$$

Therefore $\boxed{a = 4,\ b = 1}$.

(ii) $\int_0^1 \sqrt{3-2x-x^2}\,dx$

$$
\begin{aligned}
&= \int_0^1 \sqrt{p(x)}\,dx && \text{Let } x+1 = 2\sin u \qquad (*) \\
&= \int_0^1 \sqrt{4-(x+1)^2}\;\,dx && \implies \frac{dx}{du} = 2\cos u \\
&= \int_{x=0}^{x=1} \sqrt{4-4\sin^2 u}\;\,dx && x=0 \implies u = \sin^{-1}\tfrac{0+1}{2} = \tfrac{\pi}{6}, \quad \text{by } (*) \\
& && x=1 \implies u = \sin^{-1}\tfrac{1+1}{2} = \tfrac{\pi}{2} \\
&= 2\int_{u=\frac{\pi}{6}}^{u=\frac{\pi}{2}} \sqrt{1-\sin^2 u}\;\,\frac{dx}{du}\,du \\
&= 2\int_{u=\frac{\pi}{6}}^{u=\frac{\pi}{2}} \sqrt{\cos^2 u}\,(2\cos u)\,du \\
&= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 u\,du \\
&= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{2}(\cos 2u+1)\right)du \\
&= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos 2u+1)\,du \\
&= 2\left(\frac{\sin 2u}{2}+u\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= \boxed{\frac{4\pi-3\sqrt{3}}{6}}
\end{aligned}
$$

(b) We proceed using integration by parts, twice.

Let $I = \int e^{\alpha\theta} \cos\beta\theta \, d\theta$

$$
\begin{aligned}
&= uv - \int v\,du && u = e^{\alpha\theta} \quad dv = \cos\beta\theta\,d\theta \\
& && = \alpha e^{\alpha\theta}\,d\theta \quad \Rightarrow v = \frac{\sin\beta\theta}{\beta} \\
&= \frac{e^{\alpha\theta}\sin\beta\theta}{\beta} - \frac{\alpha}{\beta}\int e^{\alpha\theta}\sin\beta\theta\,d\theta \\
&= \frac{e^{\alpha\theta}\sin\beta\theta}{\beta} - \frac{\alpha}{\beta}\left(wx - \int x\,dw\right) && w = e^{\alpha\theta} \quad dx = \sin\beta\theta\,d\theta \\
& && = \alpha e^{\alpha\theta}\,d\theta \quad \Rightarrow x = -\frac{\cos\beta\theta}{\beta} \\
&= \frac{e^{\alpha\theta}\sin\beta\theta}{\beta} - \frac{\alpha}{\beta}\left(-\frac{e^{\alpha\theta}\cos\beta\theta}{\beta} + \frac{\alpha}{\beta}\int e^{\alpha\theta}\cos\beta\theta\,d\theta\right)
\end{aligned}
$$

$$
\implies I = \frac{\beta e^{\alpha\theta}\sin\beta\theta + \alpha e^{\alpha\theta}\cos\beta\theta}{\beta^2} - \frac{\alpha^2}{\beta^2} I
$$

$$
\implies \left(1 + \frac{\alpha^2}{\beta^2}\right) I = \frac{e^{\alpha\theta}(\alpha\cos\beta\theta + \beta\sin\beta\theta)}{\beta^2}
$$

$$
\implies I = \frac{e^{\alpha\theta}(\alpha\cos\beta\theta + \beta\sin\beta\theta)}{\beta^2\left(1 + \frac{\alpha^2}{\beta^2}\right)}
$$

Therefore $\int e^{\alpha\theta}\cos\beta\theta\,d\theta = \boxed{\frac{e^{\alpha\theta}}{\alpha^2+\beta^2}(\alpha\cos\beta\theta + \beta\sin\beta\theta)}$.

6. (a) If f is well-defined at $x = -2$, then $f(-2) = 1 + 2 = 3$ (since $f(x) = 1 - x$ for $x \leqslant -2$), must also equal $f(-2) = a + 2 - (-2)^2 = a - 2$ (since $f(x) = a - x - x^2$ for $-2 \leqslant x \leqslant 2$). In other words, we want the outputs at the endpoints to be the same (so that the curves are connected), thus we want $3 = a - 2 \implies a = 5$.

   Similarly at $x = 2$, we want $f(2) = 5 - 2 - 2^2 = -1$ (since $f(x) = 5 - x - x^2$ for $-2 \leqslant x \leqslant 2$) to equal $f(2) = \ln(2 + b)$ (since $f(x) = \ln(x + b)$ for $x \geqslant 2$). Thus we equate and solve $-1 = \ln(2 + b) \implies e^{-1} = 2 + b \implies b = e^{-1} - 2$.

   Therefore $\boxed{a = 5,\ b = e^{-1} - 2}$.

   (b) The sketch is given in **??**. Intercepts are determined in the usual way,

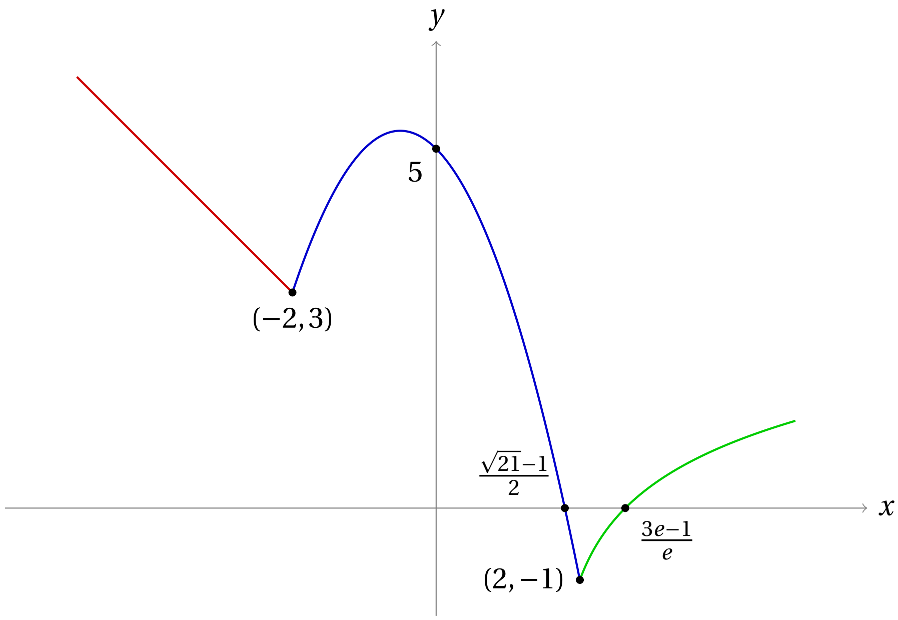


FIGURE 4: Sketch of $y = f(x)$

evaluating $f(0)$ for the f-intercept, and solving $f(x) = 0$ for the different parts of the function for the x-intercepts.

Clearly f is defined for all real numbers, so the domain $\boxed{\operatorname{dom}(f) = \mathbb{R}}$, and since no part of the curve lies below the line $f = -1$, the range $\boxed{\operatorname{ran}(f) = [-1, \infty)}$.

(c) The sketch is given in **??**. Since all of the curve lies between $y = 3$ and $y = 0$, with 0 *excluded* and 3 *included*, then $\boxed{\operatorname{ran}(g) = (0, 3]}$.

Now, to show that there are no values of x for which $f(x) = g(x)$, let us attempt to solve this equation on different parts of the common domain. When $-3 < x \leqslant -2$, the equation $f(x) = g(x)$ becomes

$$x - 1 = 3 + x,$$

where $3 - |x|$ becomes $3 + x$ since x is always negative in this part of the domain, so $|x| = -x$. The equation above clearly has no solutions. Now when $-2 \leqslant x < 0$, the equation $f(x) = g(x)$ becomes

$$5 - x - x^2 = 3 + x.$$

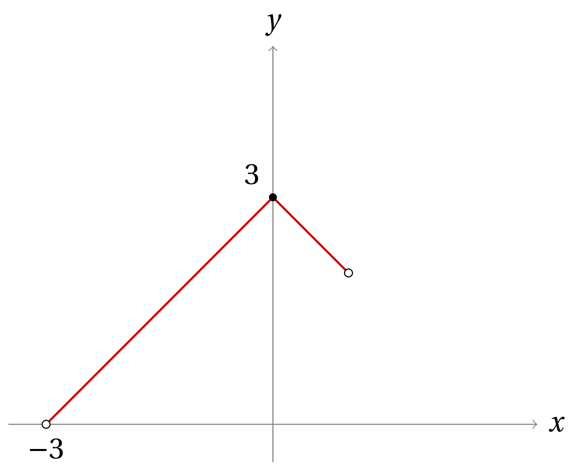


FIGURE 5: Sketch of $y = g(x)$

This is the quadratic equation $x^2 + 2x - 2 = 0$ which has solutions $x = -1 \pm \sqrt{3}$, but these are both outside the range $-2 \leqslant x < 0$, so the curves do not intersect in $-2 \leqslant x < 0$.

Finally for $0 \leqslant x < 1$, the equation $f(x) = g(x)$ becomes

$$5 - x - x^2 = 3 - x,$$

i.e., $x^2 = 2$, which has solutions $x = \pm\sqrt{2}$. These are both out of the range $0 \leqslant x < 1$, so the curves do not intersect over the range.

Thus combining our observations the curves do not intersect in the range $-3 < x < 1$, but that is the entire domain of g, so they do not intersect. □