Mock Assessment Test

LUKE'S MATHS LESSONS[*]

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Advanced Level PAPER II 25th August, 2022

Instructions

The goal of this test is to prepare you for your MATSEC advanced level pure mathematics exam. The topics assessed here are those pertaining to the paper 2 syllabus.

Read the following instructions carefully.

- This test consists of **10 questions** and carries **150 marks**.
- You have **3 hours** to complete this test.
- Attempt **7** out of the **10** questions.

[*] https://maths.mt

1. (a) (i) Solve the differential equation

$$x\frac{dy}{dx} + (1 - x\tan x)\,y = x^2,$$

given that when $x = \pi$, $y = 0$.

(b) Solve the differential equation

$$9\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 8x - 5e^{-x},$$

given that when $x = 0$, $y = \frac{14}{5}$ and $\frac{dy}{dx} = \frac{11}{5}$.
[Hint: When finding the particular integral, use $Px + Q + Re^{-x}$ as a trial solution.]

[8, 7 marks]

2. (a) Show that the equation $xe^x = \cos x$ has a solution between 0 and 1. Use the Newton–Raphson method to find an approximation for this value, taking $x = 1$ as first approximation. Do *two* iterations, and give your working to *four* decimal places.

(b) (i) Express the length of the curve $y = \sin x$ for x between 0 and π as an integral.

(ii) Estimate the integral using Simpson's rule with an interval width of $h = \frac{\pi}{6}$. Give your working to *four* decimal places.

[7, 8 marks]

3. (a) (i) Prove that

$$\log\left(1 - \frac{4}{x+2} + \frac{3}{(x+2)^2}\right) = \log(x+1) + \log(x-1) - 2\log(x+2).$$

(ii) Hence, show that

$$\sum_{r=2}^{n} \log\left(1 - \frac{4}{r+2} + \frac{3}{(r+2)^2}\right) = \log\left(\frac{18}{n(n+1)(n+2)^2}\right).$$

(b) By direct application of Maclaurin's expansion formula, show that the first two nonzero terms of the expansion of $f(x) = \log\left(\frac{1}{\sqrt{\cos x}}\right)$ are

$$\frac{x^2}{4} + \frac{x^4}{24}.$$

Hence, find

$$\lim_{x\to 0}\left(\frac{x^4}{4\log\left(\frac{1}{\sqrt{\cos x}}\right) - x^2}\right).$$

[8, 7 marks]

4. (a) Consider the matrix $\mathbf{M} = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}$. Prove by induction that for any integer $n \geqslant 0$,

$$\mathbf{M}^n = \begin{pmatrix} 1-3n & 9n \\ -n & 1+3n \end{pmatrix}.$$

(b) Using induction, show that $17^n - 2^n$ is a multiple of 5 for all $n \geqslant 0$.

(c) Show, using induction, that for $n \geqslant 5$,

$$(5\times 4) + (6\times 5) + (7\times 6) + \cdots + (n\times(n-1)) = \tfrac{1}{3}(n^3 - n - 60).$$

[5, 5, 5 marks]

5. Let $f(x) = \dfrac{6x - x^2}{x^2 - 6x + 5}$.

(a) Determine the range of values of y in which no part of the curve $y = f(x)$ exists.

(b) Using the result you obtained in part (a) or otherwise, determine the coordinates of any turning points on the curve $y = f(x)$. State also the equations of any asymptotes.

(c) Sketch the curve $y = f(x)$, clearly indicating the turning points and asymptotes found in part (b), together with the points where the curve intersects coordinate axes.

(d) Sketch the curve $y = 1/f(x)$ on a separate diagram.

[4, 4, 4, 3 marks]

6. Consider the matrix $\mathbf{A} = \begin{pmatrix} -3 & 12 & 4 \\ -2 & 7 & 2 \\ 5 & a & b \end{pmatrix}$.

   (a) Given that $(\mathbf{A} - \mathbf{I})(\mathbf{A} + 2\mathbf{I}) = \mathbf{0}_{3\times3}$, where $\mathbf{I}$ and $\mathbf{0}_{3\times3}$ are the 3×3 identity and zero matrices respectively, determine the values of a and b. Using the given equation, or otherwise, determine $\mathbf{A}^{-1}$.

   For the following question, ignore the values of a and b found in part (a).

   (b) Let $\mathbf{B}$ be the matrix $\mathbf{A}$ above, with $b = 6$.

      (i) Explain why the homogeneous equation $\mathbf{B}\boldsymbol{x} = \mathbf{0}_{3\times1}$ always has at least one solution, and that there is a unique value of a for which it has more than one solution.

      (ii) Find this value of a, and determine the solutions of the equation when a takes on this value, in the form of a vector equation. What does this vector equation represent, geometrically?

[6, 9 marks]

7. (a) Use de Moivre's Theorem to prove that for all θ,

$$32\sin^6\theta = 10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta,$$

   and determine a corresponding identity for $32\cos^6\theta$. Deduce that

$$8(\sin^6\theta + \cos^6\theta) = 5 + 3\cos 4\theta,$$

   and hence or otherwise, evaluate

$$\int_0^{64\pi} (\sin^6\theta + \cos^6\theta)^2\, d\theta.$$

   (b) Find the fifth roots of unity, and sketch them on an Argand diagram. Show that:

      (i) They can be written as $1, \omega, \omega^2, \omega^3$ and ω^4.

      (ii) Their sum is zero. [Hint: Consider $(1-\omega)(1+\omega+\omega^2+\omega^3+\omega^4)$].

      (iii) The points they represent in the complex plane form the vertices of a pentagon whose area is $\frac{5}{2}\sin\frac{2\pi}{5}$.

[8, 7 marks]

8. The limaçon curve $\mathscr{L}$ is given by the polar equation $r = 5 - 4\cos\theta$.

   (a) Use a suitable range of θ values to sketch the curve $\mathscr{L}$.

   (b) Determine the polar coordinates of the points P and Q, where the curve $\mathscr{L}$ intersects the curve $\mathscr{C}$, with equation $r = 3$. What is the curve $\mathscr{C}$?

   (c) A line passes through the point P, through the pole, and intersects the curve $\mathscr{L}$ at the point R. Determine the length of PR.

   (d) Determine the area of the region that lies within the curve $\mathscr{C}$, but outside the curve $\mathscr{L}$.

[3, 3, 3, 6 marks]

9. The position vectors of the points A, B, C and D are $\mathbf{i}+\mathbf{j}-3\mathbf{k}$, $2\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{i}-\mathbf{k}$, and $-\mathbf{i}+\mathbf{j}$ respectively. Determine:

   (a) a vector equation of the line ℓ through the points A and B,

   (b) the Cartesian equation of the plane Π_1 containing the points A, B and C,

   (c) the Cartesian equation of the plane Π_2 containing the points C and D, which does not intersect the line ℓ,

   (d) the distance of the points A and B from Π_2,

   (e) the angle between Π_1 and Π_2.

[2, 3, 4, 3, 3 marks]

10. (a) A triangle has vertices at $A(2,2)$, $B(4,4)$ and $C(6,2)$. The triangle is rotated through one complete revolution about the x-axis. Find:

    (i) the volume generated,

    (ii) the total surface area of the object formed.

    (b) A container of depth 1 metre is shaped in such a way that when the depth of water in it is x cm, then volume is $(2x^3 + x)\,\text{cm}^3$. Water is poured into the container at a constant rate of $50\,\text{cm}^3/\text{s}$. What is the rate of change of depth at the point in time when x is 40 cm?

[10, 5 marks]

Answers

1. (a) (i) By the product rule, $\frac{d}{dx}(xy\cos x) = \boldsymbol{x\cos x\frac{dy}{dx} + y\cos x - xy\sin x}$.

   (ii) The equation is $\frac{dy}{dx} + (\frac{1}{x} - \tan x)y = x$. Multiplying through by the integrating factor $\mu(x) = \exp(\int(\frac{1}{x} - \tan x)\,dx) = x\cos x$, the equation becomes $x\cos x\frac{dy}{dx} + y\cos x - xy\sin x = x^2\cos x$.

   Recognising the LHS from (i), we have $\frac{d}{dx}(xy\cos x) = x^2\cos x$, and thus integrating both sides, we get $xy\cos x = \int x^2\cos x\,dx$.

   The RHS requires integration by parts twice, and at the end we get the general solution $y(x) = \left(x - \frac{2}{x}\right)\tan x + \frac{c}{x}\sec x + 2$.

   Finally, since $y(\pi) = 0$, we get that $0 = -\frac{c}{\pi} + 2$, i.e., $c = 2\pi$. Thus the particular solution is $\boldsymbol{y(x) = \left(x - \frac{2}{x}\right)\tan x + \frac{2\pi}{x}\sec x + 2}$.

   (b) The auxiliary equation $9k^2 - 12k + 4 = 0$ has the repeated root $k = \frac{2}{3}$, thus the complementary function is $e^{2/3x}(c_1 + c_2x)$.

   Then by substituting the trial solution into the LHS of the equation, we obtain the particular integral $2x + 6 - \frac{1}{5}e^{-x}$. Putting everything together, the general solution is $y(x) = 2x + 6 - \frac{1}{5}e^{-x} + e^{2/3x}(c_1 + c_2x)$.

   Then plugging in the given information into the general solution and its derivative, we find the constants $c_1 = -3$ and $c_2 = 2$, and thus the particular solution is $\boldsymbol{y(x) = 2x + 6 - \frac{1}{5}e^{-x} + e^{2/3x}(2x - 3)}$.

2. (a) A solution corresponds to a zero of the function $f(x) = xe^x - \cos x$, which is continuous on $[0, 1]$. Now $f(0) = 0 - \cos(0) = -1 < 0$, and $f(1) = e - \cos(1) \geqslant e - 1 > 0$. Thus by the intermediate value theorem, there is some $c \in (0, 1)$ such that $f(c) = 0$.

   If we let $x_0 = 1$ be an initial approximation for the root c of f, one iteration of the Newton–Raphson formula yields $x_1 = 0.6531$, and another gives $\boldsymbol{x_2 = 0.5313}$.

   (b) (i) Using the booklet formula, $\boldsymbol{\ell = \int_0^\pi \sqrt{1 + \cos^2 x}\,dx}$.

   (ii)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\sqrt{1 + \cos^2 x}$	1.4142	1.3229	1.1180	1	1.1180	1.3229	1.4142
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Thus

$$\begin{aligned}\ell &\approx \tfrac{\pi}{18}\Big(1.4142+1.4142+4(1.3229+1+1.3229)\\ &\qquad\qquad +2(1.1180+1.1180)\quad\Big)\\ &= \mathbf{3.8194}.\end{aligned}$$

3. (a) (i) Since

$$1-\frac{4}{x+2}+\frac{3}{(x+2)^2}=\frac{(x+2)^2-4(x+2)+3}{(x+2)^2}=\frac{(x+1)(x-1)}{(x+2)^2},$$

by the laws of logarithms,

$$\begin{aligned}\log\left(1-\frac{4}{x+2}+\frac{3}{(x+2)^2}\right) &= \log\left(\frac{(x+1)(x-1)}{(x+2)^2}\right)\\ &= \log(x+1)+\log(x-1)-2\log(x+2).\end{aligned}$$

□

(ii) Thus

$$\begin{aligned}&\sum_{r=2}^{n}\log\left(1-\frac{4}{x+2}+\frac{3}{(x+2)^2}\right)\\ &= \sum_{r=2}^{n}\log(r+1)+\sum_{r=2}^{n}\log(r-1)-2\sum_{r=2}^{n}\log(r+2)\\ &= \sum_{r=3}^{n+1}\log r+\sum_{r=1}^{n-1}\log r-2\sum_{r=4}^{n+2}\log r\\ &= \log 3+\sum_{r=4}^{n-1}\log r+\log n+\log(n+1)\\ &\qquad +\log 1+\log 2+\log 3+\sum_{r=4}^{n-1}\log r\\ &\qquad -2\left(\sum_{r=4}^{n-1}\log r+\log n+\log(n+1)+\log(n+2)\right)\\ &= 2\log 3+\log 2-\log n-\log(n+1)-2\log(n+2)\\ &= \log\left(\frac{18}{n(n+1)(n+2)^2}\right).\end{aligned}$$

(b) We have

$$\begin{aligned}
f(x) &= -\tfrac{1}{2}\log(\cos x) && \implies && f(0) = 0 \\
f'(x) &= \tfrac{1}{2}\tan x && \implies && f'(0) = 0 \\
f''(x) &= \tfrac{1}{2}\sec^2 x && \implies && f''(0) = \tfrac{1}{2} \\
f'''(x) &= \sec^2 x \tan x && \implies && f'''(0) = 0 \\
f''''(x) &= \sec^4 x + 2\sec^2 x \tan^2 x && \implies && f''''(0) = 1,
\end{aligned}$$

and so by Maclaurin's theorem,

$$\begin{aligned}
f(x) &= \frac{1/2}{2!}x^2 + \frac{1}{4!}x^4 + O(x^5) \\
&= \frac{x^2}{4} + \frac{x^4}{24} + O(x^5).
\end{aligned}$$

□

Now for the desired limit, we have $\log\left(\frac{1}{\sqrt{\cos x}}\right) = \frac{x^2}{4} + \frac{x^4}{24} + O(x^5)$ as $x \to 0$, so we can substitute:

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{x^4}{4\log\left(\frac{1}{\sqrt{\cos x}}\right) - x^2}\right) &= \lim_{x\to 0}\left(\frac{x^4}{4\left(\frac{x^4}{4} + \frac{x^4}{24} + O(x^5)\right) - x^2}\right) \\
&= \lim_{x\to 0}\left(\frac{x^4}{\frac{x^4}{6} + O(x^5)}\right) \\
&= \lim_{x\to 0}\left(\frac{6}{1 + O(x)}\right) = \mathbf{6}.
\end{aligned}$$

4. (a) For the base case, with $n = 0$, it's obvious that $\mathbf{M}^0 = \mathbf{I}$ and the formula we have works. For the inductive step, we have

$$\begin{aligned}
\mathbf{M}^{k+1} = \mathbf{M}^k\mathbf{M} &\overset{\text{IH}}{=} \begin{pmatrix} 1-3k & 9k \\ -k & 1+3k \end{pmatrix}\begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} -2-3k & 9+9k \\ -1-k & 4+3k \end{pmatrix} \\
&= \begin{pmatrix} 1-3(k+1) & 9(k+1) \\ -(k+1) & 1+3(k+1) \end{pmatrix}.
\end{aligned}$$

□

(b) When $n = 0$, $17^0 - 2^0 = 0$ which is divisible by 5. For the inductive step, we have

$$\begin{aligned}
17^{k+1} - 2^{k+1} &= 17(17^k) - 2(2^k) \\
&= 17(17^k) - 17(2^k) + 15(2^k) \\
&= 17(17^k - 2^k) + 15(2^k) \\
&\overset{\text{IH}}{=} 17(5a) + 15(2^k) \\
&= 5(17a + 3 \cdot 2^k), \text{ a multiple of 5.} \qquad \square
\end{aligned}$$

(c) When $n = 5$, $5 \times 4 = 20$ and $\frac{1}{3}(5^3 - 5 - 60) = 20$, so the result holds and the base case is done.

For the inductive step,

$$\begin{aligned}
\sum_{r=5}^{k+1} r(r-1) &= \sum_{r=5}^{k} r(r-1) + (k+1)k \\
&\overset{\text{IH}}{=} \tfrac{1}{3}(k^3 - k - 60) + (k+1)k \\
&= \tfrac{1}{3}(k^3 - k - 60 + 3k^2 + 3k) \\
&= \tfrac{1}{3}(k^3 + 3k^2 + 3k + 1 - k - 1 - 60) \\
&= \tfrac{1}{3}((k+1)^3 - (k+1) - 60),
\end{aligned}$$

as required. $\square$

5. (a) Notice the equation of the curve is equivalent to

$$\begin{aligned}
& y = \frac{6x - x^2}{x^2 - 6x + 5} \\
\iff \quad & y(x^2 - 6x + 5) = 6x - x^2 \\
\iff \quad & yx^2 - 6yx + 5y = 6x - x^2 \\
\iff \quad & (y+1)x^2 - 6(y+1)x + 5y = 0 \qquad ①
\end{aligned}$$

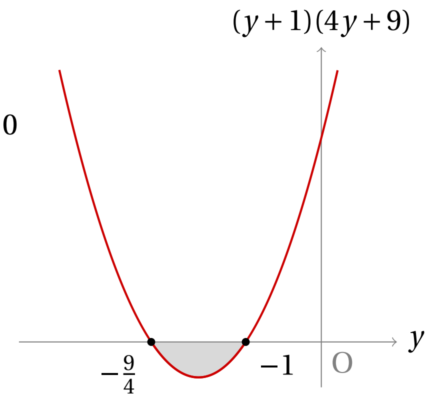
If we determine the range of values of y for which this quadratic in x has no roots, we will be finding the y-coordinates which have no associated x-coordinate; i.e., the y-coordinates where the curve does not exist.

Therefore no part of the curve exists when the quadratic discriminant $\Delta < 0$:

$$
\begin{aligned}
&36(y+1)^2 - 20y(y+1) < 0 \\
\implies &36(y^2+2y+1) - 20y^2 - 20y < 0 \\
\implies &16y^2 + 52y + 36 < 0 \\
\implies &4y^2 + 13y + 9 < 0 \\
\implies &(y+1)(4y+9) < 0
\end{aligned}
$$

From the sketch,

$\therefore\ \boldsymbol{-\frac{9}{4} < y < -1}$



(b) Since the curve exists everywhere else outside $-\frac{9}{4} < y < -1$, then any extrema must occur at one of the points where $y = -\frac{9}{4}$ and $y = -1$.

Thus, we substitute $y = -1$ and $y = -\frac{4}{9}$ in ①. Substituting $y = -1$ gives nonsense, so there are no extrema with y-coordinate -1. On the other hand, substituting $y = -\frac{9}{4}$ gives

$$
\begin{aligned}
&-\frac{5}{4}x^2 - 6\left(-\frac{5}{4}\right)x + 5\left(-\frac{9}{4}\right) = 0 \\
\implies &5x^2 - 30x + 45 = 0 \\
\implies &x^2 - 6x + 9 = 0 \\
\implies &(x-3)^2 = 0 \\
\implies &x = 3 \quad \text{(twice)}
\end{aligned}
$$

Therefore a turning point occurs at $\left(\mathbf{3}, -\frac{\mathbf{9}}{\mathbf{4}}\right)$.

Now, to determine asymptotes. Vertical asymptotes occur when the denominator is zero:

$$
\begin{aligned}
&x^2 - 6x + 5 = 0 \\
\implies &(x-1)(x-5) = 0 \\
\implies &x = 1 \quad \text{or} \quad x = 5
\end{aligned}
$$

Therefore the equations of the vertical asymptotes are $\boldsymbol{x = 1}$ and $\boldsymbol{x = 5}$.

Horizontal or oblique asymptotes occur as $x \to \pm\infty$. We therefore determine an asymptotic formula for f as x gets large:

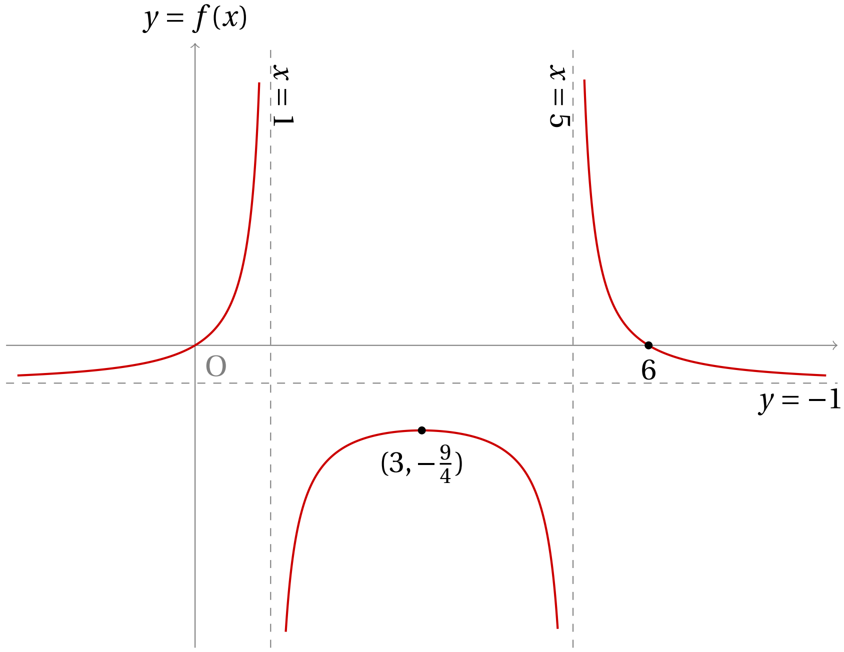
$$f(x) = \frac{6x - x^2}{x^2 - 6x + 5} = -1 + \frac{5}{x^2 - 6x + 5} = -1 + O\left(\frac{1}{x^2}\right)$$

as $x \to \pm\infty$. In other words, as x gets big, $f(x) \sim -1$, and so $\boldsymbol{y = -1}$ is a horizontal asymptote to the curve.

(c) For the y-intercept, set $x = 0 \implies f(0) = 0$.

For the x-intercepts, we solve $f(x) = 0$, which happens when the numerator is zero: $6x - x^2 = 0 \implies x(6 - x) = 0 \implies x = 0$ or $x = 6$.

Sketch:

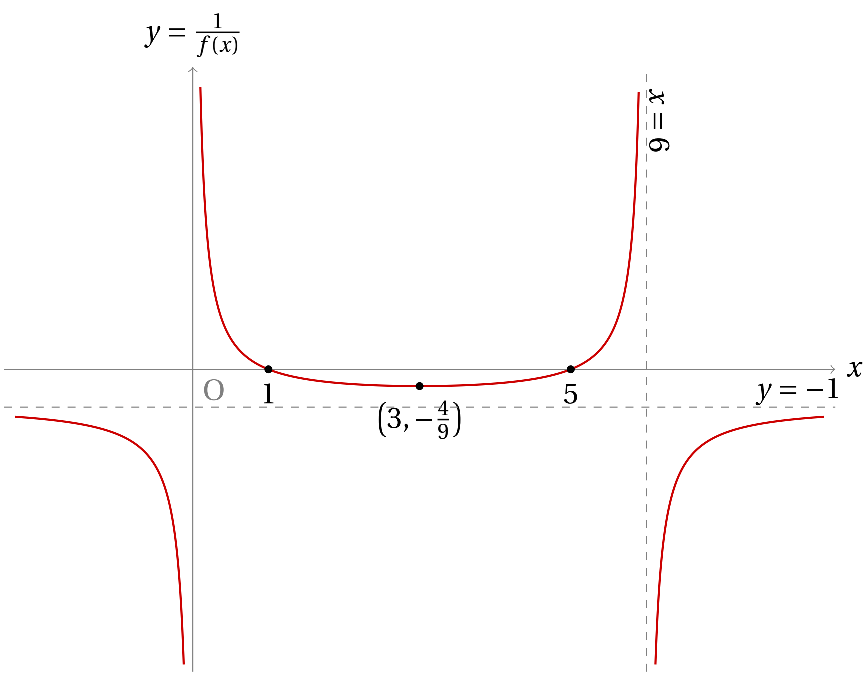


(d) For the graph of $y = 1/f(x)$, we make the following considerations.

- Any x-intercepts of the curve $y = f(x)$ are roots of $y = 1/f(x)$ and vice-versa. (So now, we have asymptotes at $x = 0$ and $x = 6$, and roots at $x = 1$ and $x = 5$).
- Each respective part of the curve $y = f(x)$ remains in the same quadrant when considering the curve $y = 1/f(x)$.

- If $f(x) \to \infty$, then $1/f(x) \to 0^+$ (from above) and vice-versa. Similarly, if $f(x) \to -\infty$, then $1/f(x) \to 0^-$ (from below) and vice-versa.
- If $y = f(x)$ has a maximum turning point at (x_0, y_0), then $y = 1/f(x)$ has a minimum turning point at $\left(x_0, \frac{1}{y_0}\right)$ and vice-versa. (So now, a minimum turning point occurs at $\left(3, -\frac{4}{9}\right)$).
- Any horizontal asymptotes given by $y = a$ are still present in the curve $y = 1/f(x)$, however they are shifted to $y = \frac{1}{a}$. (The asymptote at $y = -1$ remains at $y = -1$).

With these in mind, we can proceed to sketch $y = 1/f(x)$.



6. (a) Substituting $\mathbf{A}$ in the LHS,

$$
\begin{aligned}
&(\mathbf{A}-\mathbf{I})(\mathbf{A}+2\mathbf{I}) \\
&= \left[\begin{pmatrix} -3 & 12 & 4 \\ -2 & 7 & 2 \\ 5 & a & b \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \left[\begin{pmatrix} -3 & 12 & 4 \\ -2 & 7 & 2 \\ 5 & a & b \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right] \\
&= \begin{pmatrix} -4 & 12 & 4 \\ -2 & 6 & 2 \\ 5 & a & b-1 \end{pmatrix} \begin{pmatrix} -1 & 12 & 4 \\ -2 & 9 & 2 \\ 5 & a & b+2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 4a+60 & 4(b+2)+8 \\ 0 & 2a+30 & 2(b+2)+4 \\ -2a+5(b-1)-5 & (b-1)a+9a+60 & 2a+(b-1)(b+2)+20 \end{pmatrix}.
\end{aligned}
$$

Comparing entries with the zero matrix, we can quickly see that we have $\boldsymbol{a} = \mathbf{-15}$ **and** $\boldsymbol{b} = \mathbf{-4}$.

Now we can use the equation to determine $\mathbf{A}^{-1}$:

$$
\begin{aligned}
& & (\mathbf{A}-\mathbf{I})(\mathbf{A}+2\mathbf{I}) &= \mathbf{0} \\
&\implies & \mathbf{A}^2+\mathbf{A}-2\mathbf{I} &= \mathbf{0} \\
&\implies & \mathbf{A}^{-1}(\mathbf{A}^2+\mathbf{A}-2\mathbf{I}) &= \mathbf{A}^{-1}\mathbf{0} \\
&\implies & \mathbf{A}+\mathbf{I}-2\mathbf{A}^{-1} &= \mathbf{0} \\
&\implies & \mathbf{A}^{-1} &= \tfrac{1}{2}(\mathbf{A}+\mathbf{I}) \\
&\therefore & \mathbf{A}^{-1} &= \mathbf{\frac{1}{2}} \begin{pmatrix} \mathbf{-2} & \mathbf{12} & \mathbf{4} \\ \mathbf{-2} & \mathbf{8} & \mathbf{2} \\ \mathbf{5} & \mathbf{-15} & \mathbf{-3} \end{pmatrix}
\end{aligned}
$$

(b) (i) The trivial solution $\mathbf{x} = (0,0,0)$ is always a valid solution to the system.

Now we perform Gaussian elimination on the augmented matrix $(\mathbf{B}|\mathbf{0})$.

$$
(\mathbf{B}|\mathbf{0}) = \left(\begin{array}{ccc|c} -3 & 12 & 4 & 0 \\ -2 & 7 & 2 & 0 \\ 5 & a & 6 & 0 \end{array}\right)
$$

$2R_1 + (-3)R_2 \to R_2$
$5R_1 + 3R_3 \to R_3$

$$\sim \left(\begin{array}{ccc|c} -3 & 12 & 4 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 60+3a & 38 & 0 \end{array}\right)$$

$(-19)R_2 + R_3 \to R_3$

$$\sim \left(\begin{array}{ccc|c} -3 & 12 & 4 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 3+3a & 0 & 0 \end{array}\right)$$

Observe that the third row corresponds to $0x+(3+3a)y+0z = 0$. If we take $a = -1$, then the equation (trivially) has infinitely many solutions. If $a \neq -1$, then we must have $y = 0$, and consequently, the other rows will give us $x = z = 0$. Therefore only the value $a = -1$ gives more than one solution. □

(ii) From part (i), $\boldsymbol{a} = \mathbf{-1}$. Now from R_2, we get $3y+2z = 0$, which rearranges to $z = -\frac{3}{2}y$. Similarly, from R_1, we get $-3x+12y+4z = 0$. Making y subject from the previous equation and substituting gives $-3x + 12(-\frac{2}{3}z) + 4z = 0 \implies -3x - 8z + 4z = 0 \implies z = -\frac{3}{4}x$.

Thus combining these equations:

$$-\frac{3}{4}x = -\frac{3}{2}y = z$$
$$\implies \frac{x-0}{-\frac{4}{3}} = \frac{y-0}{-\frac{2}{3}} = \frac{z-0}{1}$$

This corresponds to the Cartesian equation $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ of a line in $\mathbb{R}^3$ with initial point (x_1, y_1, z_1) and direction vector (a, b, c).

The corresponding vector equation is $\mathbf{r} = \boldsymbol{\lambda}(\mathbf{4i} + \mathbf{2j} - \mathbf{3k})$. This line represents the intersection of the three planes $-3x + 12y + 4z = 0$, $-2x + 7y + 2z = 0$, $5x - y + 6z = 0$.

7. (a) Recall the following consequence of de Moivre's theorem: if we write

$w = e^{i\theta}$, then $w^n + w^{-n} = 2\cos n\theta$ and $w^n - w^{-n} = 2i\sin n\theta$. Thus,

$$\begin{aligned}
(2i\sin\theta)^6 &= \left(w - w^{-1}\right)^6 \\
\implies -64\sin^6\theta &= w^6 - 6w^4 + 15w^2 - 20 + 15w^{-2} - 6w^{-4} + w^{-6} \\
&= (w^6 + w^{-6}) - 6(w^4 + w^{-4}) + 15(w^2 + w^{-2}) - 20 \\
&= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20 \\
\therefore \quad 32\sin^6\theta &= 10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta
\end{aligned}$$

We proceed similarly to obtain the identity for $32\cos^6\theta$:

$$\begin{aligned}
(2\cos\theta)^6 &= \left(w + w^{-1}\right)^6 \\
\implies 64\cos^6\theta &= w^6 + 6w^4 + 15w^2 + 20 + 15w^{-2} + 6w^{-4} + w^{-6} \\
&= (w^6 + w^{-6}) + 6(w^4 + w^{-4}) + 15(w^2 + w^{-2}) + 20 \\
&= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20 \\
\therefore \quad 32\cos^6\theta &= 10 + 15\cos 2\theta + 6\cos 4\theta + \cos 6\theta.
\end{aligned}$$

Adding the two identities, we get $32\sin^6\theta + 32\cos^6\theta = 20 + 12\cos 4\theta$, which reduces to $8(\sin^6\theta + \cos^6\theta) = 5 + 3\cos 4\theta$, as required.

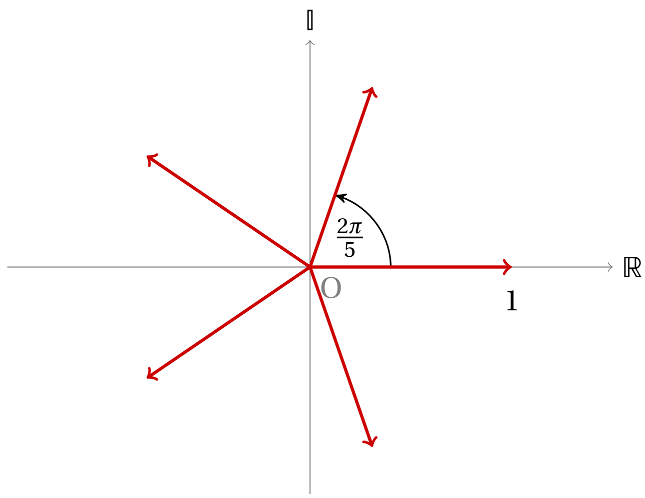
Therefore the desired integral becomes

$$\begin{aligned}
&\int_0^{64\pi} (\sin^6\theta + \cos^6\theta)^2\, d\theta \\
&= \frac{1}{64}\int_0^{64\pi} (5 + 3\cos 4\theta)^2\, d\theta \\
&= \frac{1}{64}\int_0^{64\pi} (25 + 30\cos 4\theta + 9\cos^2 4\theta)\, d\theta \\
&= \frac{1}{64}\left(\left[25\theta + \frac{15}{2}\sin 4\theta\right]_0^{64\pi} + \frac{9}{2}\int_0^{64\pi} (\cos 8\theta + 1)\, d\theta\right) \\
&= \left[\frac{25\theta}{64} + \frac{15\sin 4\theta}{128} + \frac{9\sin 8\theta}{1024} + \frac{9\theta}{128}\right]_0^{64\pi} \\
&= 25\pi + \frac{9\pi}{2} = \boldsymbol{\frac{59\pi}{2}}.
\end{aligned}$$

(b) The fifth roots of unity are the solutions to the equation $z^5 = 1$. Clearly, each solution has $|z| = \sqrt[5]{1} = 1$. Thus all we have to determine are the different possible values of $\arg z$. We know that the possible values of $\arg z$, where $z^n = re^{i\alpha}$, are given by

$$\theta = \frac{2k\pi \pm \alpha}{n}, \quad k \in \mathbb{Z},$$

so long as $-\pi < \theta \leqslant \pi$. In our case, we have $\alpha = 0$ and $n = 1$, so the different possible values are $\theta = \left\{-\frac{4\pi}{5}, -\frac{2\pi}{5}, 0, \frac{2\pi}{5}, \frac{4\pi}{5}\right\}$. Therefore the fifth roots of unity are $\mathbf{1, e^{\pm 2\pi i/5}, e^{\pm 4\pi i/5}}$.



(i) Let $\omega = e^{2\pi i/5}$. Then by de Moivre's theorem, $\omega^2 = \left(e^{2\pi i/5}\right) = e^{4\pi i/5}$, which is another root. Similarly, $\omega^3 = e^{6\pi i/5}$, again by de Moivre's theorem. Now $\frac{6\pi}{5}$ is out of the range $-\pi < \theta \leqslant \pi$, but as an angle, it is equivalent to $\frac{6\pi}{5} - 2\pi = -\frac{4\pi}{5}$, therefore ω^3 equals $e^{-4\pi i/5}$. Finally, $\omega^4 = e^{8\pi i/5}$, whose argument is equivalent to $\frac{8\pi}{5} - 2\pi = \frac{-2\pi}{4}$, the remaining value of θ.

Therefore the roots can be expressed as $1, \omega, \omega^2, \omega^3$ and ω^4. $\square$

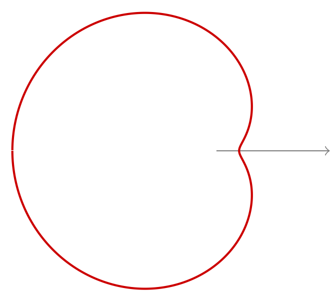
(ii) $(1-\omega)(1+\omega+\omega^2+\omega^3+\omega^4) = 1-\omega^5 = 0$ by definition of the the fifth roots of unity. Thus either $1-\omega = 0$, i.e. $\omega = 1$, or $1+\omega+\omega^2+\omega^3+\omega^4 = 0$. Taking $\omega = 1$ will not give us a meaningful result in this case; so we may discard the former. Taking ω as in part (i), we get that the sum of all the fifth roots of unity is zero. $\square$

(iii) If we refer to the diagram drawn in part (i), the pentagon is obtained simply by joining the vertices together, giving rise to 5 isosceles triangles, each with apex angle $\frac{2\pi}{5}$ and legs of length 1. Thus the area is $5\times\frac{1}{2}ab\sin C = \frac{5}{2}(1)(1)\sin\frac{2\pi}{5} = \mathbf{\frac{5}{2}\sin\frac{2\pi}{5}}$, as required. □

8. (a) Since r is a function of $\cos\theta$, it suffices to take θ in the range $0\leqslant\theta\leqslant\pi$, since $\cos\theta$ is an even function and negative angles would give the same result.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	1	1.54	2.17	3	5	7	7.83	8.46	9

Sketch:



(b) We solve $5-4\cos\theta = 3$ to find the values of θ at which points of intersection occur.

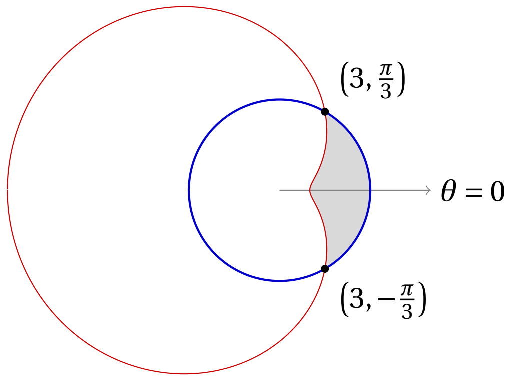
$$\begin{aligned} & 5-4\cos\theta = 3 \\ \implies \quad & \cos\theta = \tfrac{1}{2} \\ \implies \quad & \theta_{\text{p.v.}} = \cos^{-1}\left(\tfrac{1}{2}\right) = \tfrac{\pi}{3} \\ \implies \quad & \theta = 2n\pi \pm \tfrac{\pi}{3}, \qquad n\in\mathbb{Z} \end{aligned}$$

Taking values of n other than $n=0$ gives values outside the range $-\pi<\theta\leqslant\pi$, thus the only values of θ where intersection points occur are $\theta=\pm\frac{\pi}{3}$, and so the points of intersection are $\boldsymbol{P}=\left(\mathbf{3},\frac{\pi}{3}\right)$ **and** $\boldsymbol{Q}=\left(\mathbf{3},-\frac{\pi}{3}\right)$.

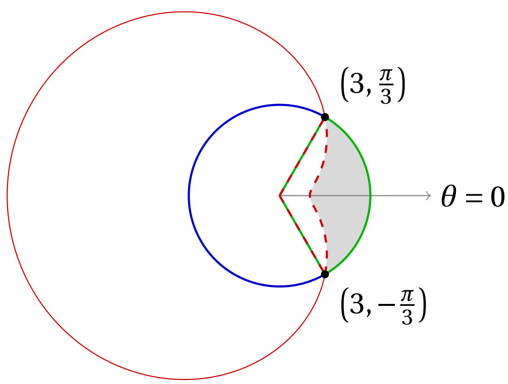
The curve $\mathscr{C}$ represents a **circle**, since it has a fixed radius of $r=3$ independent of the angle θ.

(c) A line through P then the pole makes an angle of $\frac{\pi}{3}-\pi=-\frac{2\pi}{3}$. Thus we evaluate $5-4\cos\left(-\frac{2\pi}{3}\right)=7$, which represents the distance from the pole to the point R. Therefore the distance PR is the distance from the pole to P ($r = 3$) plus the distance from the pole to R ($r = 7$), i.e. $|PR| = \mathbf{10}$.

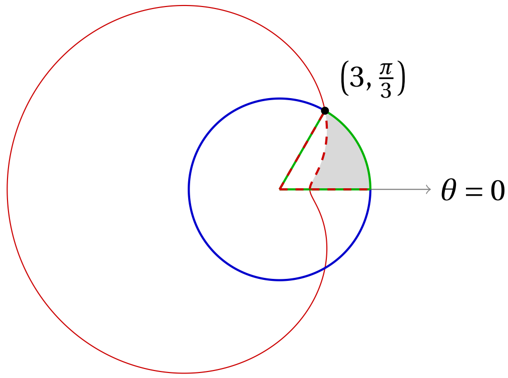
(d) The desired area is the following:



We know that in general, the area enclosed by the curve $r = r(\theta)$ and the part-lines $\theta = a$ and $\theta = b$ is given by $\frac{1}{2}\int_a^b r^2\, d\theta$. Now our desired area is the area of the circle between $r = \frac{\pi}{3}$ (outlined in green) and $r = -\frac{\pi}{3}$, minus that of the curve $\mathscr{L}$ in that region (outlined in red, dashed):



Furthermore, since the region is symmetric in the horizontal, we can simply evaluate the following enclosed region instead, then multiply the result by two.



Therefore,

$$\begin{aligned}
A &= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (3)^2\, d\theta - \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (5-4\cos\theta)^2\, d\theta \\
&= \int_0^{\frac{\pi}{3}} 9\, d\theta - \int_0^{\frac{\pi}{3}} (25 - 40\cos\theta + 16\cos^2\theta)\, d\theta \\
&= 9\theta - 25\theta + 40\sin\theta\Big|_0^{\frac{\pi}{3}} - 8\int_0^{\frac{\pi}{3}} (1+\cos 2\theta)\, d\theta \\
&= 20\sqrt{3} - \frac{16\pi}{3} - 8\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_0^{\frac{\pi}{3}} \\
&= \mathbf{18\sqrt{3} - 8\pi}\ \text{units}^{\mathbf{2}}.
\end{aligned}$$

9. (a) Take $\mathbf{a} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$ as the initial point, and $\vec{AB} = \mathbf{b} - \mathbf{a} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ as its direction. Thus ℓ_1 has equation $\mathbf{r = i + j - 3k + \lambda(i - 3j + 4k)}$.

(b) Since the points A, B and C lie on Π_1, then the vectors $\vec{AB} = \mathbf{b} - \mathbf{a} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\vec{AC} = \mathbf{c} - \mathbf{a} = -\mathbf{j} + 2\mathbf{k}$ lie in Π_1. Therefore we can take $\mathbf{n}_1 = \vec{AB} \times \vec{AC}$ to be the normal of Π_1, where

$$\begin{aligned}
\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 4 \\ 0 & -1 & 2 \end{vmatrix} &= \begin{vmatrix} -3 & 4 \\ -1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -3 \\ 0 & -1 \end{vmatrix}\mathbf{k} \\
&= -2\mathbf{i} - 2\mathbf{j} - \mathbf{k}
\end{aligned}$$

Therefore Π_1 has vector equation $\mathbf{r} \cdot \mathbf{n}_1 = \mathbf{a} \cdot \mathbf{n}_1$, i.e. $\mathbf{r} \cdot (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = (\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$, which simplifies to $\mathbf{r} \cdot (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 1$, having the corresponding Cartesian equation $\mathbf{2x + 2y + z = 1}$.

(c) Since Π_2 contains the points C and D, then the vector $\vec{CD} = \mathbf{d} - \mathbf{c} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$ lie on Π_2. Furthermore, since Π_2 does not intersect ℓ, then it must be parallel to ℓ, i.e., its direction vector $\vec{AB} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ lies in the plane Π_2. Therefore we can define $\mathbf{n}_2 = \vec{AB} \times \vec{CD}$ to be the normal of Π_2, where

$$\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 4 \\ -2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -3 & 4 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ -2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} \mathbf{k}$$
$$= -7\mathbf{i} - 9\mathbf{j} - 5\mathbf{k}$$

Therefore Π_2 has vector equation $\mathbf{r} \cdot \mathbf{n}_2 = \mathbf{c} \cdot \mathbf{n}_2$, i.e., $\mathbf{r} \cdot (7\mathbf{i} + 9\mathbf{j} + 5\mathbf{k}) = (\mathbf{i} - \mathbf{k}) \cdot (7\mathbf{i} + 9\mathbf{j} + 5\mathbf{k})$, which simplifies to $\mathbf{r} \cdot (7\mathbf{i} + 9\mathbf{j} + 5\mathbf{k}) = 2$, having the corresponding Cartesian equation $\boldsymbol{7x + 9y + 5z = 2}$.

(d) The distance of a point X, with position vector $\mathbf{x}$, from a plane $\Pi : \mathbf{r} \cdot \mathbf{n} = d$ is given by the formula $s = \left| \mathbf{x} \cdot \hat{\mathbf{n}} - \frac{d}{\|\mathbf{n}\|} \right|$. In the case of Π_2, we have $\|\mathbf{n}_2\| = \sqrt{7^2 + 9^2 + 5^2} = \sqrt{155}$, so $\hat{\mathbf{n}}_2 = \frac{7}{\sqrt{155}}\mathbf{i} + \frac{9}{\sqrt{155}}\mathbf{j} + \frac{5}{\sqrt{155}}\mathbf{k}$. Thus for the point A, we have:

$$\left| \mathbf{a} \cdot \hat{\mathbf{n}}_2 - \frac{2}{\|\mathbf{n}_2\|} \right| = \left| \frac{7}{\sqrt{155}} + \frac{9}{\sqrt{155}} - \frac{15}{\sqrt{155}} - \frac{2}{\sqrt{155}} \right| = \frac{\mathbf{1}}{\sqrt{\mathbf{155}}} \text{ units}$$

Similarly, for point B, we have

$$\left| \mathbf{b} \cdot \hat{\mathbf{n}}_2 - \frac{2}{\|\mathbf{n}_2\|} \right| = \left| \frac{14}{\sqrt{155}} - \frac{18}{\sqrt{155}} + \frac{5}{\sqrt{155}} - \frac{2}{\sqrt{155}} \right| = \frac{\mathbf{1}}{\sqrt{\mathbf{155}}} \text{ units}$$

(e) The angle θ between two planes whose normals are $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by the formula $\cos\theta = \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2$. In our case, we have $\hat{\mathbf{n}}_1 = \frac{\mathbf{n}_1}{\|\mathbf{n}_1\|} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ and $\hat{\mathbf{n}}_2 = \frac{7}{\sqrt{155}}\mathbf{i} + \frac{9}{\sqrt{155}}\mathbf{j} + \frac{5}{\sqrt{155}}\mathbf{k}$, thus

$$\begin{aligned} \cos\theta &= \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 \\ &= \frac{14}{3\sqrt{155}} + \frac{18}{3\sqrt{155}} + \frac{5}{3\sqrt{155}} \\ &= \frac{37}{3\sqrt{115}} \\ \implies \theta &= \cos^{-1}\left(\frac{37}{3\sqrt{115}}\right) \approx \mathbf{7.84}^\circ \end{aligned}$$

10. (a)