TELESCOPING SERIES

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Certain series have general terms which may be written as a difference of two terms of the same shape, and consequently exhibit lots of cancellation. For instance, the sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)}$$

does not immediately look like there is cancellation, but if we expand the general term into partial fractions, we see that actually it equals

$$\sum_{k=1}^{n} \Big(\frac{1}{k} - \frac{1}{k+1}\Big),$$

whose general term is a difference of the form $f(k) - f(k+1)$. Thus the terms are

$$\Big(\frac{1}{1} - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \cdots + \Big(\frac{1}{n-1} - \frac{1}{n}\Big) + \Big(\frac{1}{n} - \frac{1}{n+1}\Big),$$

and rebracketing, we have

$$\frac{1}{1} + \Big(-\frac{1}{2} + \frac{1}{2}\Big) + \Big(-\frac{1}{3} + \frac{1}{3}\Big) + \cdots + \Big(-\frac{1}{n} + \frac{1}{n}\Big) - \frac{1}{n+1},$$

so all the middle terms cancel and we are left with the nice formula

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}.$$

Sums which exhibit such cancellation are called **telescoping sums**. (Think of the terms cancelling as equivalent to the act of collapsing a telescope.)

Remark. Notice that we can also infer the sum to infinity

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} = \lim_{n\to\infty} \Big(1 - \frac{1}{n+1}\Big) = 1.$$

Working with Σ-Notation

It might make you feel uneasy that we are relying on the ellipsis-form $(\cdots)$ of the series to carry out this manipulation. In truth, we can work entirely with Σ-notation instead, and this actually makes things easier when the cancellation is not as straightforward as in the example we just gave.

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We simply need to make two observations about how we can manipulate sums. Observe that we can shift the indices of a sum by any whole number X:

$$\sum_{k=a}^{b} f(k) = \sum_{k=a+X}^{b+X} f(k-X).$$

Indeed, both sums represent $f(a) + \cdots + f(b)$. (A good way to check that you did this correctly when working is to think of what the first and last terms are before and after the manipulation, and to make sure that they agree. For instance,

$$\sum_{k=1}^{n} \frac{1}{k+2} = \sum_{k=3}^{n+2} \frac{1}{k},$$

both have first term $\frac{1}{3}$ and last term $\frac{1}{n+2}$.) The second observation we make is that we can extract any number of terms from a sum, either at the start, or at the end, so long as we adjust the indices appropriately; for instance, we can take out three terms at the start of

$$\sum_{k=1}^{n} f(k)$$

to get

$$f(1) + f(2) + f(3) + \sum_{k=4}^{n} f(k),$$

or the first two and last two to get

$$f(1) + f(2) + \sum_{k=3}^{n-2} f(k) + f(n-1) + f(n).$$

By combining these two techniques, we can make the cancellation obvious in Σ-notation. Indeed, the example we just gave amounts to rewriting the series as

$$\begin{aligned}
\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) &= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} \\
&= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k} \\
&= 1 + \sum_{k=2}^{n} \frac{1}{k} - \sum_{k=2}^{n} \frac{1}{k} - \frac{1}{n+1} \\
&= 1 - \frac{1}{n+1}.
\end{aligned}$$

So in summary, the two steps we need to perform to evaluate a telescoping sum are:

(1) split the sum using linearity, and adjust the indices of summation so that all sums present have identical general terms;
(2) take out terms from the start and end of each sum so that you are left with identical sums which cancel.

Here is another example: observe that we can work out $\sum_{k=1}^{n} \cos k\theta$ by writing the general term $\cos k\theta$ as a difference: if we multiply it by $\sin\theta$, we can use the identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ to get

$$\sin\theta\cos k\theta = \tfrac{1}{2}(\sin(k\theta+\theta) + \sin(\theta - k\theta)) = \tfrac{1}{2}(\sin(k+1)\theta - \sin(k-1)\theta),$$

i.e.,

$$\cos k\theta = \tfrac{1}{2}\csc\theta(f(k+1) - f(k-1))$$

where $f(k) = \sin k\theta$. Thus

$$\begin{aligned}\sum_{k=1}^{n}\cos k\theta &= \frac{\csc\theta}{2}\Big(\sum_{k=1}^{n}\sin(k+1)\theta - \sum_{k=1}^{n}\sin(k-1)\theta\Big)\\ &= \frac{\csc\theta}{2}\Big(\sum_{k=2}^{n+1}\sin k\theta - \sum_{k=0}^{n-1}\sin k\theta\Big)\\ &= \frac{\csc\theta}{2}\Big(\sum_{k=2}^{n-1}\sin k\theta + \sin n\theta + \sin(n+1)\theta - \sin\theta - \sum_{k=2}^{n-1}\sin k\theta\Big)\\ &= \tfrac{1}{2}\csc\theta\big(\sin(n\theta) + \sin(n+1)\theta - \sin\theta\big),\end{aligned}$$

which is a nice closed form.

Exercise.

(1) Show that

$$\sum_{k=2}^{n}\frac{1}{k^2-1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)},$$

and deduce the value of $\sum_{k=2}^{\infty}\frac{1}{k^2-1}$.

(2) Show that the closed form we obtained for $\sum_{k=1}^{n}\cos k\theta$ can be further simplified to

$$\csc\tfrac{\theta}{2}\sin n\tfrac{\theta}{2}\cos(n+1)\tfrac{\theta}{2}.$$

(3) Similarly prove that

$$\sum_{k=1}^{n}\sin k\theta = \csc\tfrac{\theta}{2}\sin n\tfrac{\theta}{2}\sin(n+1)\tfrac{\theta}{2}.$$

Generality of the Method

Notice that it isn't necessarily the case that the general term is of the form $f(k)-f(k+1)$. In fact, any combination $a_1 f(k+b_1) + a_2 f(k+b_2) + \cdots + a_s f(k+b_s)$ of a shifted term $f(k)$ with coefficients $a_1, \ldots, a_s$ such that $a_1 + a_2 + \cdots + a_s = 0$ is a candidate for this method. For instance, take the terms

$$\sqrt{k+2}, \qquad \sqrt{k-1}, \quad \text{and} \quad \sqrt{k},$$

and combine them with coefficients $2, -3$ and 1 (since these add up to zero), so that we have the sum

$$\sum_{k=1}^{n} 2\sqrt{k+2} - 3\sqrt{k-1} + \sqrt{k}.$$

We can apply the method as usual:

$$\begin{aligned}\sum_{k=1}^{n} 2\sqrt{k+2} - 3\sqrt{k-1} + \sqrt{k} &= 2\sum_{k=1}^{n}\sqrt{k+2} - 3\sum_{k=1}^{n}\sqrt{k-1} + \sum_{k=1}^{n}\sqrt{k} \\ &= 2\sum_{k=3}^{n+2}\sqrt{k} - 3\sum_{k=0}^{n-1}\sqrt{k} + \sum_{k=1}^{n}\sqrt{k} \\ &= 2\Big(\sum_{k=3}^{n-1}\sqrt{k} + \sqrt{n} + \sqrt{n+1} + \sqrt{n+2}\Big) \\ &\quad - 3\Big(1 + \sqrt{2} + \sum_{k=3}^{n-1}\sqrt{k}\Big) + 1 + \sqrt{2} + \sum_{k=3}^{n-1}\sqrt{k} + \sqrt{n} \\ &= 3\sqrt{n} + 2\sqrt{n+1} + 2\sqrt{n+2} - 2 - 2\sqrt{2},\end{aligned}$$

and notice that precisely because the coefficients add up to zero, we have that the remaining identical sums all cancel, leaving us with a closed form for the sum.

Example (MATSEC May 2020, Question 4). Find

$$S_n = \sum_{k=1}^{n} \frac{5k+13}{(k+2)(k+3)(k+5)},$$

and deduce the sum to infinity S_∞.

Solution. Expanding partial fractions, we have

$$S_n = \sum_{k=1}^{n}\Big(\frac{1}{k+2} + \frac{1}{k+3} - \frac{2}{k+5}\Big).$$

Notice that these are all translated versions of the same general term $f(k) = \frac{1}{k}$, and that the coefficients $1, 1, -2$ sum to zero. This tells us that we have a telescoping sum.

$$\begin{aligned}S_n &= \sum_{k=1}^{n}\frac{1}{k+2} + \sum_{k=1}^{n}\frac{1}{k+3} - 2\sum_{k=1}^{n}\frac{1}{k+5} \\ &= \sum_{k=0}^{n-1}\frac{1}{k+3} + \sum_{k=1}^{n}\frac{1}{k+3} - 2\sum_{k=3}^{n+2}\frac{1}{k+3} \\ &= \Big(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \sum_{k=3}^{n-1}\frac{1}{k+3}\Big) + \Big(\frac{1}{4} + \frac{1}{5} + \sum_{k=1}^{n-1}\frac{1}{k+3} + \frac{1}{n+3}\Big) \\ &\quad - 2\Big(\sum_{k=3}^{n-1}\frac{1}{k+3} + \frac{1}{n+3} + \frac{1}{n+4} + \frac{1}{n+5}\Big) \\ &= \frac{37}{30} - \frac{1}{n+3} - \frac{2}{n+4} - \frac{2}{n+5}.\end{aligned}$$

As $n \to \infty$, the terms in n all go to zero, so we have that $S_\infty = 37/30$. □