L-Università ta' Malta

Department of Technology and
Entrepreneurship Education
Faculty of Education

B.Sc. (Hons.) in Technical Design and Technology — Year I

May/June Examination Session 2022

TET1013: Calculus for Technology

23rd June 2022
10:00–13:05

Instructions

Read the following instructions carefully.

- The mark you obtain in this examination carries **80%** of the final mark associated with this study-unit.
- Attempt only **FIVE** questions from Section A and only **FIVE** questions from Section B.
- Each question carries **10** marks. The maximum mark is **100**.
- A list of mathematical formulae is provided on page 2.
- Only the use of non-programmable calculators is allowed.

MATHEMATICAL FORMULÆ

ALGEBRA

Factors

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Quadratics

If $ax^2 + bx + c$ has roots α and β,

$$\Delta = b^2 - 4ac$$
$$\alpha + \beta = -\tfrac{b}{a} \qquad \alpha\beta = \tfrac{c}{a}$$

Finite Series

$$\sum_{k=1}^{n} 1 = n \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
$$\sum_{k=1}^{n} k^2 = \frac{k(k+1)(2k+1)}{6}$$
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^n$$
$$= 1 + nx + \tfrac{n(n-1)}{2\cdot 1} x^2 + \cdots + x^n$$

GEOMETRY & TRIGONOMETRY

Distance Formula

If $A = (x_1, y_1)$ and $B = (x_2, y_2)$,

$$d(A,B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{\Delta x^2 + \Delta y^2}$$

Pythagorean Identity

$$\cos^2\theta + \sin^2\theta = 1$$

General Solutions

$$\cos\theta = \cos\alpha \iff \theta = \pm\alpha + 2\pi\mathbb{Z}$$
$$\sin\theta = \sin\alpha \iff \theta = (-1)^n\alpha + \pi n,\ n \in \mathbb{Z}$$
$$\tan\theta = \tan\alpha \iff \theta = \alpha + \pi\mathbb{Z}$$

CALCULUS

Derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
e^x	e^x
$\log x$	$1/x$
uv	$u'v + uv'$
u/v	$(u'v - uv')/v^2$

Integrals

$f(x)$	$\int f(x)\,dx$
$x^n\ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$\log(\sec x)$
$\cot x$	$\log(\sin x)$
$\sec x$	$\log(\sec x + \tan x)$
$\operatorname{cosec} x$	$\log(\tan \frac{x}{2})$
e^x	e^x
$1/x$	$\log x$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\frac{x}{\sqrt{a^2+x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$

Homogeneous Linear Second Order ODEs

If the roots of $ak^2 + bk + c$ are k_1 and k_2, then the differential equation $ay'' + by' + cy = 0$ has general solution

$$y(x) = \begin{cases} c_1 e^{k_1 x} + c_2 e^{k_2 x} & \text{if } k_1 \neq k_2 \\ c_1 e^{kx} + c_2 x e^{kx} & \text{if } k = k_1 = k_2 \\ e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) & \text{if } k = \alpha \pm \beta i \in \mathbb{C} \end{cases}$$

Integral Formulae

For a curve $y = f(x)$ with $a \leqslant x \leqslant b$,

$$\text{Area between curve and } x\text{-axis} = \int_a^b f(x)\,dx$$
$$\text{Mean value of } f = \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\text{Volume of revolution about } x\text{-axis} = \pi\int_a^b [f(x)]^2\,dx$$
$$\text{Length of arc} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$
$$\text{Surface area of revolution about } x\text{-axis} = 2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

SECTION A

⚠ Attempt only **FIVE** questions from this section.

A1. The function f is defined by

$$f(x) = 2x^3 - 7x^2 + 9.$$

(a) Find $f'(x)$ from first principles, using the limit definition.

(b) Hence show that when x is around 2, then $f(x) \approx 5 - 4x$.

(c) Find the coordinates of the turning points on the curve $y = f(x)$.

[4, 3, 3 marks]

A2. (a) Find the derivatives of the following functions.

(i) $x^2 \cos 2x$ (ii) $\sqrt{1 + e^{x^2}}$ (iii) $\log\left(\dfrac{\sqrt[3]{2+x}}{x^2\sqrt{1+x}}\right)$

(b) Verify that $y = \sin(e^x)$ is a solution to the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + e^{2x}y = 0.$$

[7, 3 marks]

A3. Consider the curve given by the equation

$$y = \frac{3+4x}{1+x^2}.$$

(a) Determine the coordinates of stationary points on the curve.

(b) Determine their nature.

(c) Sketch the curve, labelling any turning points and intercepts with the x- and y-axes.

[4, 3, 3 marks]

A4. Determine the following integrals.

(a) $\displaystyle\int_0^1 x\sqrt{x}\,dx$ (b) $\displaystyle\int \sin(2x-1)\,dx$

(c) $\displaystyle\int \frac{3x^2-5x}{(x^2+x+1)(2x-1)}\,dx$

[2, 2, 6 marks]

A5. (a) Sketch the graphs $y = x^2+1$ and $y = 3-x^2$ on the same set of axes.

(b) Show that the equations of the tangents to the curve $y = x^2+1$ at the points where it intersects the other curve are $y = \pm 2x$.

(c) Find the area bounded by the two curves.

[3, 3, 4 marks]

A6. Solve the differential equation

$$(x^2+x-2)\frac{dy}{dx} = 3y,$$

given that $y = -\frac{1}{2}$ when $x = 0$. Give your solution in the form

$$y(x) = \frac{x+a}{x+b},$$

where a and b are constants.

[10 marks]

A7. Solve the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x^2,$$

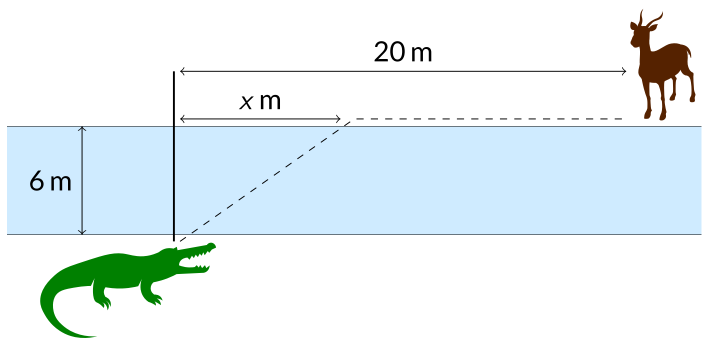
given that when $x = 0$, y and $\frac{dy}{dx}$ are both 1.

[10 marks]

SECTION B

⚠ Attempt only **FIVE** questions from this section.

B1. A crocodile is stalking a gazelle that is 20 m upstream on the opposite side of a river. In water, crocodiles travel at 4 m/s, whereas on land, they travel at 5 m/s. Suppose the crocodile swims to a point that is x m upstream on the opposite bank of the river, and runs on land the rest of the way, as depicted below.



(a) Show that the time taken for the crocodile to reach the gazelle is given by

$$T(x) = \frac{\sqrt{36+x^2}}{4} + \frac{20-x}{5}.$$

(b) What is the time taken if the crocodile does not travel on land?

(c) What is the time taken if it swims the shortest distance possible?

(d) What is x if the crocodile gets to the gazelle as fast as possible?

[4, 1, 1, 4 marks]

B2. (a) A piece of string has the form of the curve $y = (2x+7)^{3/2}$ from $x = 0$ to $x = 2$. Show that the length of the string is $488/27$.

(b) The product of two positive numbers is the same as their average. What is the least possible value for the logarithm of the sum of their squares?

[5, 5 marks]

B3. A brick is in the shape of a cuboid with base x cm by $2x$ cm, and height h cm. The total surface area of the brick is $300\,\text{cm}^2$.

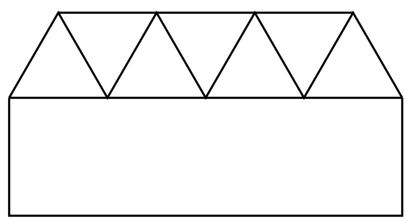
(a) Show that $h = \dfrac{50}{x} - \dfrac{2x}{3}$.

(b) The volume of the brick is $V\,\text{cm}^3$. Express V in terms of x only.

(c) Find by differentiation the maximum value of V.

[3, 3, 4 marks]

B4. An enclosure is to be constructed using wooden fencing. Seen from above, the enclosure resembles the following shape, made from seven equilateral triangles surmounting a rectangle.



The total amount of fencing used will be 120 m.

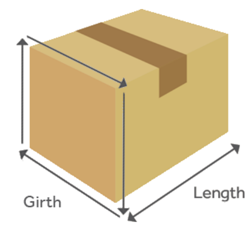
(a) If the side length of a single equilateral triangle is x, show that the total area enclosed is

$$A(x) = 240x - \left(38 - \tfrac{7\sqrt{3}}{4}\right)x^2.$$

(b) Hence, determine the maximum possible area, accurate to three significant figures.

[6, 4 marks]

B5. Maltapost regulations state that the sum of the length and girth of a postal package cannot exceed 2 m in total.



Determine:

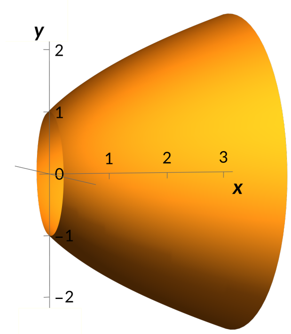
(a) The volume of the largest package with a square base that can be sent by post.

(b) The volume of the largest cylindrical package that can be sent by post.

(c) Carnival is close and a Maltese friend is stranded abroad with no access to *perlini* (i.e., traditional small sweets). Which of the two packages is best for the emergency supply so that he gets as many as possible?

[4, 4, 2 marks]

B6. Consider the curve $y = \sqrt{2x+1}$.



(a) Sketch the curve in the range $0 \leqslant x \leqslant 3$.

(b) A video game designer made a digital model of a lampshade by rotating the part of the curve in the range $0 \leqslant x \leqslant 3$ around the x-axis.

Find:

(i) The volume enclosed by the lampshade,

(ii) The surface area of the exterior of the lampshade.

[2, 3, 5 marks]