L-Università ta' Malta

Department of Technology and Entrepreneurship Education
Faculty of Education

B.Sc. (Hons.) Year I

Sample Examination Paper I

TET1013: Calculus for Technology

*n*th June 20XX
10:00–13:05

Instructions

Read the following instructions carefully.

- The mark you obtain in this examination carries **80%** of the final mark associated with this study-unit.
- Attempt only **FIVE** questions from Section A and only **FIVE** questions from Section B.
- Each question carries **10** marks. The maximum mark is **100**.
- A list of mathematical formulae is provided on page 2.
- Only the use of non-programmable calculators is allowed.

MATHEMATICAL FORMULÆ

ALGEBRA

Factors

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Quadratics

If $ax^2 + bx + c$ has roots α and β,

$$\Delta = b^2 - 4ac$$
$$\alpha + \beta = -\tfrac{b}{a} \qquad \alpha\beta = \tfrac{c}{a}$$

Finite Series

$$\sum_{k=1}^{n} 1 = n \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
$$\sum_{k=1}^{n} k^2 = \frac{k(k+1)(2k+1)}{6}$$
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^n$$
$$= 1 + nx + \tfrac{n(n-1)}{2\cdot 1}x^2 + \cdots + x^n$$

GEOMETRY & TRIGONOMETRY

Distance Formula

If $A = (x_1, y_1)$ and $B = (x_2, y_2)$,

$$d(A,B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{\Delta x^2 + \Delta y^2}$$

Pythagorean Identity

$$\cos^2\theta + \sin^2\theta = 1$$

General Solutions

$$\cos\theta = \cos\alpha \iff \theta = \pm\alpha + 2\pi\mathbb{Z}$$
$$\sin\theta = \sin\alpha \iff \theta = (-1)^n\alpha + \pi n,\ n \in \mathbb{Z}$$
$$\tan\theta = \tan\alpha \iff \theta = \alpha + \pi\mathbb{Z}$$

CALCULUS

Derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
e^x	e^x
$\log x$	$1/x$
uv	$u'v + uv'$
u/v	$(u'v - uv')/v^2$

Integrals

$f(x)$	$\int f(x)\,dx$
$x^n\ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$\log(\sec x)$
$\cot x$	$\log(\sin x)$
$\sec x$	$\log(\sec x + \tan x)$
$\operatorname{cosec} x$	$\log(\tan \frac{x}{2})$
e^x	e^x
$1/x$	$\log x$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\frac{x}{\sqrt{a^2+x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$

Homogeneous Linear Second Order ODEs

If the roots of $ak^2 + bk + c$ are k_1 and k_2, then the differential equation $ay'' + by' + cy = 0$ has general solution

$$y(x) = \begin{cases} c_1 e^{k_1 x} + c_2 e^{k_2 x} & \text{if } k_1 \neq k_2 \\ c_1 e^{kx} + c_2 x e^{kx} & \text{if } k = k_1 = k_2 \\ e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) & \text{if } k = \alpha \pm \beta i \in \mathbb{C} \end{cases}$$

Integral Formulae

For a curve $y = f(x)$ with $a \leqslant x \leqslant b$,

$$\text{Area between curve and } x\text{-axis} = \int_a^b f(x)\,dx$$
$$\text{Mean value of } f = \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\text{Volume of revolution about } x\text{-axis} = \pi\int_a^b [f(x)]^2\,dx$$
$$\text{Length of arc} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$
$$\text{Surface area of revolution about } x\text{-axis} = 2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

SECTION A

⚠ Attempt only **FIVE** questions from this section.

A1. Consider the curve given by the equation

$$y = \frac{7e^{-x}}{4x^2+3}.$$

(a) Determine the coordinates of stationary points on the curve.

(b) Determine their nature.

(c) Sketch the curve, labelling any turning points and intercepts with the x- and y-axes.

[3, 4, 3 marks]

A2. (a) Find the derivatives of the following functions.

(i) $\log\left(\dfrac{\sqrt[3]{2x-1}}{x^{2/5}\sqrt{x-1}}\right)$ (ii) $\sqrt{x}(\log 2x)^2$ (iii) $\dfrac{x}{\sqrt{1+\cos^2 x}}$

(b) Verify that the function $y = \sqrt{x}e^{x^2}$ is a solution to the differential equation

$$4x^2\frac{d^2y}{dx^2} - 8x\frac{dy}{dx} + (5-16x^4)y = 0.$$

[6, 4 marks]

A3. Solve the differential equation

$$(x+1)(x+2)\frac{dy}{dx} = \cos^2 y,$$

given that $y = 0$ when $x = 0$. Give your solution in the form

$$y(x) = \tan^{-1}\left(\log\left(\frac{a(1+x)}{2+x}\right)\right),$$

where a is a constant to be determined. **[10 marks]**

A4. Consider the functions $f(x) = 4 - x^2$ and $g(x) = x^2 + x + 1$.

(a) Sketch $y = f(x)$ and $g(x)$ on the same set of axes. Label the points where the two points intersect.

(b) Find the equation of the tangent to $y = f(x)$ at the point where $x = 1$, and find where it intersects $y = g(x)$ again.

(c) Find the area bounded by the two curves.

[3, 3, 4 marks]

A5. Determine the following integrals.

(a) $\displaystyle\int_1^2 \frac{\sqrt{x}(x+1)}{\sqrt[3]{x}}\,dx$

(b) $\displaystyle\int \sec^2(2x-1)\,dx$

(c) $\displaystyle\int \frac{x}{(x^2+1)(x+1)}\,dx$

(d) $\displaystyle\int_0^2 \frac{3x+1}{(x+3)(x^2+3x+4)}\,dx$

[2, 1, 3, 4 marks]

A6. Solve the differential equation

$$4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = e^x,$$

given that when $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$.

[10 marks]

A7. The function f is defined by $f(x) = \sqrt{5 + x^2}$.

(a) Show that

$$\frac{f(x+h) - f(x)}{h} = \frac{2x + h}{\sqrt{5 + (x+h)^2} + \sqrt{5 + x^2}}$$

Hence, find $f'(x)$ from first principles, using the limit definition.

(b) Hence show that when x is around 2, then $f(x) \approx \frac{1}{3}(2x + 5)$.

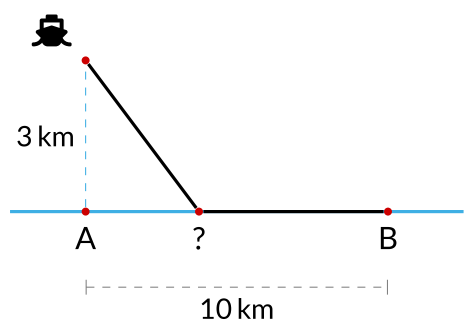
(c) Find the coordinates of the turning points on the curve $y = f(x)$.

[5, 3, 2 marks]

SECTION B

⚠ Attempt only **FIVE** questions from this section.

B1. A man rows 3 km out to sea from a starting point A. He wishes to get to a point B as fast as possible, 10 km away from A down the coast.



He can row 4 km an hour and run 5 km an hour. Assuming the coast is straight, determine:

(a) How long it would take him if he rows to A and then runs to B.

(b) How long it would take him if he rows directly to B.

(c) How far he should land from A so that he arrives in the shortest time possible.

[2,2,6 marks]

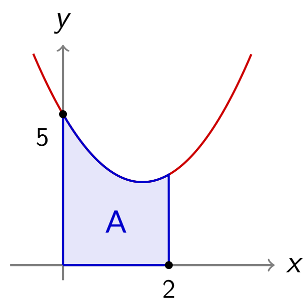
B2. The company *Tankijiet tal-Bejt Ltd.* supplies closed cylindrical water tanks of volume $4\,\text{m}^3$.

(a) What should the radius of the base be to minimise the amount of plastic used for the production of the tank?

(b) A workman suggests that using a cuboid shaped tank with a square base would minimise the amount of plastic even further. Is the workman correct?

[5, 5 marks]

B3. Consider the function $f(x) = x^2 - 3x + 5$, whose graph is sketched below.



(a) Show that the integral

$$\int \sqrt{1+(ax+b)^2}\,dx$$

equals

$$\frac{1}{2a}\left[(b+ax)\sqrt{1+(ax+b)^2} + \log\left(ax+b+\sqrt{1+(ax+b)^2}\right)\right] + c.$$

[*Hint*: use differentiation.]

(b) Find the area of the shape A.

(c) Find the perimeter of the shape A.

[5, 2, 3 marks]

B4. (a) Two industrial chimneys A and B, 100 m apart, are polluting the surrounding environment. It is estimated that the pollution level P, in parts per million, at a distance x m measured from chimney A towards chimney B, is given by $P = x^2 - 120x + 3700$.

(i) Find the value of x where the pollution level is at a minimum.

(ii) A local resident claims it is safer to live right next to chimney A than 15 m from chimney A (towards B). Is this true?

(b) Two positive numbers x and y add up to 200. What is the largest possible value of xy^2?

[6, 4 marks]

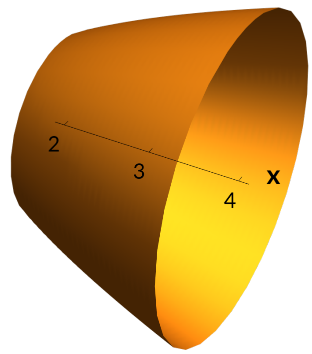
B5. Consider the curve $y = \sqrt{x-1}$.

(a) Sketch the curve in the range $2 \leqslant x \leqslant 4$.

(b) A computer model of a bowl is made by rotating the part of the curve in the range $2 \leqslant x \leqslant 4$ through one complete revolution about the x-axis.
Find:

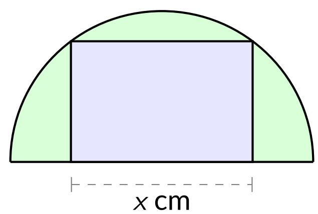
(i) The volume of the bowl,

(ii) The surface area of the exterior of the bowl.



[2, 3, 5 marks]

B6. A rectangle is drawn inside a semicircle of radius 10 cm such that one if its sides, of length x cm, is along the diameter.



(a) Show that the area of the rectangle is

$$A(x) = \tfrac{x}{2}\sqrt{400 - x^2},$$

(b) Find the dimensions of the rectangle with the largest possible area.

(c) What is the green area in that case?

[4, 4, 2 marks]