L-Università ta' Malta

Department of Technology and Entrepreneurship Education
Faculty of Education

B.Sc. (Hons.) Year I

Sample Examination Paper II

TET1013: Calculus for Technology

*n*th June 20XX
10:00–13:05

Instructions

Read the following instructions carefully.

- The mark you obtain in this examination carries **80%** of the final mark associated with this study-unit.
- Attempt only **FIVE** questions from Section A and only **FIVE** questions from Section B.
- Each question carries **10** marks. The maximum mark is **100**.
- A list of mathematical formulae is provided on page 2.
- Only the use of non-programmable calculators is allowed.

MATHEMATICAL FORMULÆ

ALGEBRA

Factors

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Quadratics

If $ax^2 + bx + c$ has roots α and β,

$$\Delta = b^2 - 4ac$$
$$\alpha + \beta = -\tfrac{b}{a} \qquad \alpha\beta = \tfrac{c}{a}$$

Finite Series

$$\sum_{k=1}^{n} 1 = n \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
$$\sum_{k=1}^{n} k^2 = \frac{k(k+1)(2k+1)}{6}$$
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^n$$
$$= 1 + nx + \tfrac{n(n-1)}{2\cdot 1} x^2 + \cdots + x^n$$

GEOMETRY & TRIGONOMETRY

Distance Formula

If $A = (x_1, y_1)$ and $B = (x_2, y_2)$,

$$d(A,B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{\Delta x^2 + \Delta y^2}$$

Pythagorean Identity

$$\cos^2\theta + \sin^2\theta = 1$$

General Solutions

$$\cos\theta = \cos\alpha \iff \theta = \pm\alpha + 2\pi\mathbb{Z}$$
$$\sin\theta = \sin\alpha \iff \theta = (-1)^n\alpha + \pi n,\ n \in \mathbb{Z}$$
$$\tan\theta = \tan\alpha \iff \theta = \alpha + \pi\mathbb{Z}$$

CALCULUS

Derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
e^x	e^x
$\log x$	$1/x$
uv	$u'v + uv'$
u/v	$(u'v - uv')/v^2$

Integrals

$f(x)$	$\int f(x)\,dx$
$x^n\ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$\log(\sec x)$
$\cot x$	$\log(\sin x)$
$\sec x$	$\log(\sec x + \tan x)$
$\operatorname{cosec} x$	$\log(\tan \frac{x}{2})$
e^x	e^x
$1/x$	$\log x$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\frac{x}{\sqrt{a^2+x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$

Homogeneous Linear Second Order ODEs

If the roots of $ak^2 + bk + c$ are k_1 and k_2, then the differential equation $ay'' + by' + cy = 0$ has general solution

$$y(x) = \begin{cases} c_1 e^{k_1 x} + c_2 e^{k_2 x} & \text{if } k_1 \neq k_2 \\ c_1 e^{kx} + c_2 x e^{kx} & \text{if } k = k_1 = k_2 \\ e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) & \text{if } k = \alpha \pm \beta i \in \mathbb{C} \end{cases}$$

Integral Formulae

For a curve $y = f(x)$ with $a \leqslant x \leqslant b$,

$$\text{Area between curve and } x\text{-axis} = \int_a^b f(x)\,dx$$
$$\text{Mean value of } f = \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\text{Volume of revolution about } x\text{-axis} = \pi \int_a^b [f(x)]^2\,dx$$
$$\text{Length of arc} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$
$$\text{Surface area of revolution about } x\text{-axis} = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

SECTION A

⚠ Attempt only **FIVE** questions from this section.

A1. Solve the differential equation

$$2\sqrt{x}\,\frac{dy}{dx} = (y^2+1)(x+1)(5x-2),$$

given that $y = 0$ when $x = 0$. Express your solution in the form

$$y(x) = \tan(\sqrt{x}(ax^2+bx+c)).$$

[10 marks]

A2. Solve the differential equation

$$y'' - 4y' + 5y = 36e^{5x},$$

given that when $x = 0$, y and y' are both 0.

[10 marks]

A3. (a) Find the derivatives of the following functions.

(i) $\log\left(\dfrac{\sqrt[3]{2x}\,\sqrt[4]{4x^3}}{\sqrt{x^2-1}}\right)$ (ii) $e^x\tan(x^2)$ (iii) $\dfrac{\cos(x^2)}{\sqrt{1+\sin(x^2)}}$

(b) Verify that $y(x) = e^{2x^2}$ is a solution to the second order differential equation $xy'' = y' + 16x^3y$.

[6, 4 marks]

A4. (a) Sketch the graphs of $y = x^2 - 1$ and $y = 2 + x - x^2$ on the same set of axes.

(b) Show that the tangents to the curve $y = 2 - x - x^2$ at the points where it intersects the other curve meet at $(\frac{1}{4}, \frac{15}{4})$.

(c) Find the area bounded by the two curves.

[3, 3, 4 marks]

A5. Consider the curve given by the equation

$$y = \frac{(x-1)^3}{\sqrt{e^x}}$$

(a) Determine the coordinates of stationary points on the curve.

(b) Determine their nature.

(c) Sketch the curve, labelling any turning points and intercepts with the x- and y-axes.

[3, 4, 3 marks]

A6. Consider the function f, defined by

$$f(x) = x^3 - 12x + 16.$$

(a) Find $f'(x)$ from first principles, using the limit definition.

(b) Show that when x is close to 3, $f(x) \approx 15x - 38$.

(c) Find the coordinates of the turning points on the curve $y = f(x)$.

[4, 3, 3 marks]

A7. Find the following integrals.

(a) $\displaystyle\int \frac{4x-5}{(x^2-x+1)(2x+1)}\,dx$

(b) $\displaystyle\int_0^{\pi} \cos\left(\frac{\theta-\pi}{6}\right) d\theta$

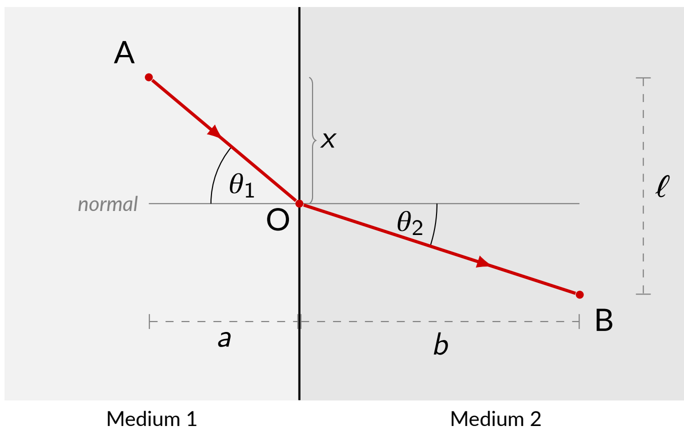
(c) $\displaystyle\int_2^{2\sqrt{3}} \frac{x+1}{x(x^2+4)}\,dx$

[3, 3, 4 marks]

SECTION B

⚠ Attempt only **FIVE** questions from this section.

B1. Consider a ray of light passing from one medium to another.



Given that light passes through medium 1 with velocity v_1, and through medium 2 with velocity v_2, show that the total time taken to travel from A to B through the variable point O is

$$T(x) = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (\ell - x)^2}}{v_2}.$$

Fermat's principle states that light travels the path which takes the least time. Assuming this principle, deduce *Snell's law*:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

[10 marks]

B2. A wire of total length 100 cm is cut into two pieces. The first piece is bent into an equilateral triangle of side length x cm, the second piece is bent into a square.

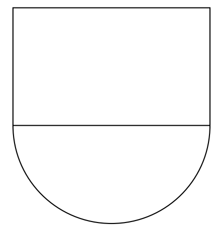
(a) Show that the combined total area of the two shapes is

$$A(x) = \tfrac{\sqrt{3}}{4}x^2 + \left(25 - \tfrac{3}{4}x\right)^2.$$

(b) Find the maximum possible combined area.

[5, 5 marks]

B3. The figure on the right depicts a design of a theatre stage, which is in the shape of a semicircle attached to a rectangle.



If the total perimeter of the stage is 60 m, show that the maximum possible area of the stage is

$$\frac{1800}{\pi+4}\,\text{m}^2.$$

[10 marks]

B4. (a) A wire is in the shape of the curve $y = 2x\sqrt{x}$ for $0 \leqslant x \leqslant 7$. Find the length of the wire.

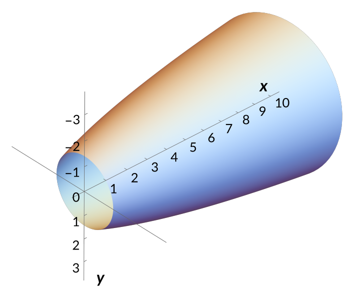
(b) The product of two numbers is equal to their average. What is the least value of the logarithm of their sum?

[5, 5 marks]

B5. Consider the curve $y = \sqrt{x+2}$.

(a) Sketch the curve in the range $0 \leqslant x \leqslant 10$.

(b) A megaphone has the shape obtained by rotating the part of the curve in the range $0 \leqslant x \leqslant 10$ around the x-axis.



Find:

(i) The volume enclosed by the megaphone,

(ii) The surface area of the exterior of the megaphone.

[2, 3, 5 marks]

B6. A tissue paper box must have a volume of $144\ \text{cm}^3$, and two of the vertical sides must be squares.

If the material for the square sides costs twice as much as the rest (because of the folding and overlap), what are the dimensions of the cheapest box?

[10 marks]